Reduced Basis Model Reduction for Nonlinear Evolution Equations based on Empirical Operator Interpolation

Martin Drohmann*, Bernard Haasdonk†, Mario Ohlberger ‡
*Sandia National Laboratories, Livermore, CA, USA
†Institute for Applied Analysis and Numerical Simulation, Stuttgart, Germany
‡Institute for Computational and Applied Mathematics, Münster, Germany

***Abstract*— Many important problems from chemistry, physics or biology are modeled by nonlinear partial differential equations. Often, e.g. in the context of inverse problems, optimization or uncertainty quantification, the numerical models need to be evaluated for many different parameter values. As numerical simulations consume a huge amount of computational power, *many–query* applications are still infeasible to deal with.**

The reduced basis can be a remedy for such situations. It features an automatic procedure to generate projection based reduced order models equipped with reliable error bounds, which are efficiently computable. The latter allows to control the accuracy to an extent that matches the reliability of the underlying high dimensional discretization.

In this presentation, we present a variant of this method that can be applied to a variety of nonlinear evolution equations. Numerical examples are provided for various partial differential equations from the field of computational fluid dynamics.

I. Introduction

The reduced basis method is a means to deal with *many–query* or *real–time* applications based on simulations of partial differential equations. The idea is to let the experimenter choose a parametrization of the problem that restricts the manifold of possible solutions $\mathcal{S}$ to those that are "interesting" to the application. Then, the essential characteristics of this manifold can be extracted as basis functions of a low dimensional *reduced basis space.* The generating algorithms depend on few solutions of a high–dimensional model, but give rise to efficient and still reliable reduced models. Reference [4] provides a good overview of the machinery with a focus on linear problems.

For nonlinear problems, it is not sufficient to approximate the solution manifold $\mathcal{S}$ by a linear space. Supplementary, the nonlinear behaviour of the system needs to be "learnt" and approximated by the so–called *empirical operator interpolation* method.

In this presentation, we want to focus on nonlinear parametrized evolution equations of the form

$$\partial_t u(\boldsymbol{\mu}; t) + \mathcal{L}(\boldsymbol{\mu}) u(\boldsymbol{\mu}; t) = 0, \qquad u(\boldsymbol{\mu}; 0) = u_0(\boldsymbol{\mu}) \quad (1)$$

TABLE I

Exemplary run–time comparison.

N	M	ø-run–time[s]	max. error	offline time[h]
$H = 7200$	–	90.01	0.00	0
42	72	4.44	$1.73 \cdot 10^{-3}$	0.54
83	144	6.04	$5.74 \cdot 10^{-5}$	1.09
125	216	8.37	$7.30 \cdot 10^{-6}$	1.55
167	288	11.92	$7.63 \cdot 10^{-7}$	2.08
208	360	15.08	$2.31 \cdot 10^{-7}$	2.69
233	402	16.48	$1.55 \cdot 10^{-7}$	3.27

defined on a regular domain Ω, and supplemented with adequate boundary conditions. Here, $\boldsymbol{\mu}$ denotes a vector of parameters, controlling e.g. material constants, the domain geometry, boundary or initial conditions. For the numerical model, we want to focus on finite volume discretizations by an Euler scheme of the form

$$\left(\mathrm{Id} + \Delta t \mathcal{L}_h^I(\boldsymbol{\mu})\right) u_h^{k+1}(\boldsymbol{\mu}) = \left(\mathrm{Id} - \Delta t \mathcal{L}_h^E(\boldsymbol{\mu})\right) u_h^k(\boldsymbol{\mu}), \quad (2)$$

where the operator $\mathcal{L}$ is decomposed into implicit and explicit computations. After the generation of a suitable reduced basis and empirical interpolants for the operators, all parameter independent parts can be reduced by a projection onto the reduced basis space. This allows to compute (2) with low computational complexity.

Table I shows time gains and accuracy results from one of our experiments with a nonlinear diffusion problem. Compared to the high dimensional problem, the time gain factor is in the range of 8-20 depending on the size of reduced basis N and the number of interpolation points M.

In the following, we want to dwell shortly into the concepts of basis construction, empirical operator interpolation and a posteriori error estimation.

II. Greedy basis construction

If the experimenter defines the parameter vector $\boldsymbol{\mu}$ and can choose very tight constraints on the possible set of vectors $\mathcal{M}$, he gets a rather small response surface of interesting solutions $\mathcal{S} := \{u_h(\boldsymbol{\mu}) | \boldsymbol{\mu} \in \mathcal{M}\}$. Then, we can assume that a low–dimensional reduced basis exists spanning a linear space that (almost) comprises this manifold.

The theoretical measure for the suitability of the manifold $\mathcal{S}$ is given by the Kolmogorov N-width of this manifold.

$$d_N(\mathcal{S}) := \inf_{\substack{\mathcal{V} \\ \dim(\mathcal{V})=N}} \sup_{v_h \in \mathcal{S}} \min_{u_h \in \mathcal{V}} \|v_h - u_h\|_{\mathcal{W}_h}. \quad (3)$$

Though not explicitly computable, [3] made the following observation: If the Kolmogorov N-width converges to zero with growing N at polynomial or exponential speed, we can construct sequences of reduced basis spaces that follow this convergence. This is achieved by iterative extension with a "greedy" algorithm: Given a reduced basis space of dimension N, a new basis vector can be defined by

$$\phi_{N+1} = \arg \min_{u_h(\boldsymbol{\mu}) \in \mathcal{S}} \|u_h(\boldsymbol{\mu}) - u_{\text{red}}(\boldsymbol{\mu})\|, \quad (4)$$

where the reduced solutions $u_{\text{red}}(\boldsymbol{\mu})$ are computed with a low–dimensional reduced basis version of (2). Instead of computing the true error, however, we want to use a posteriori error estimates that can be efficiently computed. This allows to search the manifold $\mathcal{S}$ for a larger set of new potential reduced basis functions.

III. Empirical operator interpolation

The concept applied here, is based on the empirical interpolation method (EIM) [2]. The method empirically learns about interpolation points and ansatz functions for the solutions of interest. This gives rise to interpolations

$$\mathcal{I}_M[\mathcal{L}_h]v_h = (\mathcal{L}_h v_h)\,|_{X_M} \Xi_M^t \approx \mathcal{L}_h v_h. \quad (5)$$

Here Ξ_M denotes a matrix of interpolation ansatz functions, and $(v)_{X_M}$ refers to the vector of function evaluations $(v(x_1), \ldots, v(x_M))$ at interpolation points $X_M := \{x_1, \ldots, x_M\}$. In order to efficiently compute the operator evaluations at these interpolation points, the operator needs to fulfill some sparsity properties. This, however, is usually true for numerical schemes.

IV. A posteriori error estimates

From our discussion about the greedy algorithms to construct the reduced basis space, it follows that efficiently computable error estimates are crucial ingredients for reduced basis methods. In this presentation, we obtain bounds $\eta^k_{N,M,M'}(\boldsymbol{\mu})$, such that

$$\left\|u_h^k(\boldsymbol{\mu}) - u_{\text{red}}^k(\boldsymbol{\mu})\right\| \leq \eta^k_{N,M,M'}(\boldsymbol{\mu}). \quad (6)$$

The subscripts of the error refer to the dimension N of the reduced basis space, the number of interpolation points M, and a further parameter M' controlling the accuracy of the computed error induced by the empirical operator interpolation.

Figure 1 shows convergence results of a greedy algorithm with different a posteriori error estimates, proving their suitability for this problem.

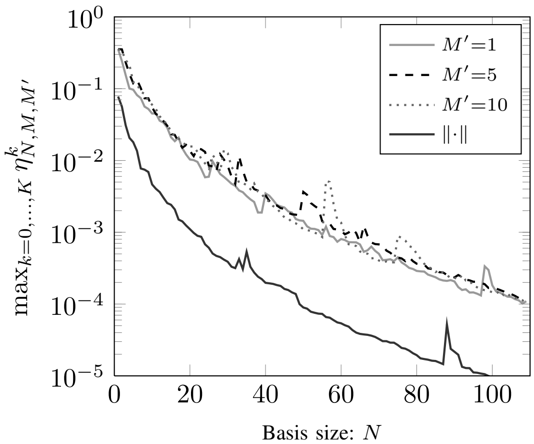

Fig. 1. Comparison of estimated error decrease during basis generation with basis generation algorithm. Different error indicators are used in order to select the worst approximated trajectories.

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