Characterizing Reduced-Order Manifolds by Finite-Time Lyapunov Analysis

Marco Maggia*, Kenneth D. Mease*
*University of California Irvine / Mechanical and Aerospace Engineering, Irvine, CA, USA

***Abstract*— Finite-time Lyapunov analysis (FTLA) is used to determine reduced-order manifold structures in the flow of nonlinear dynamical systems exhibiting two-timescale behavior. The approach is described and explained in the context of a simple example.**

I. Introduction

The purpose of the paper is to present a methodology for diagnosing two-timescale nonlinear behavior and characterizing the associated manifold structure, and to illustrate its use. The methodology is inspired by the intrinsic low-dimensional manifold method (ILDM) [1], but instead of using eigenvalues and eigenvectors, it uses finite-time Lyapunov exponents and vectors. The methodology [2], [3] derives from the asymptotic theory of partially hyperbolic systems [4]. The use of Lyapunov exponents and vectors has also been explored in [5].

II. Finite-Time Lyapunov Analysis

The goal of our analysis is to diagnose two-timescale behavior of nonlinear dynamical systems and to determine points on the reduced-order slow manifold. The timescale information is based on the finite-time Lyapunov exponents and vectors (FTLE/Vs). Consider a nonlinear dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with associated tangent linear dynamics $\dot{\mathbf{v}} = D\mathbf{f}(\mathbf{x})\mathbf{v}$ where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{v}(t) \in T_{\mathbf{x}}\mathbb{R}^n$. $\Phi(t, \mathbf{x})$ denotes the fundamental matrix of the linear dynamics for initial conditions $\mathbf{x}(0) = \mathbf{x}$ and $\Phi(0, \mathbf{x}) = I_n$. The forward and backward FTLEs are given by

$$\mu^{\pm}(T, \mathbf{x}, \mathbf{v}) := (\ln \|\Phi(\pm T, \mathbf{x})\mathbf{v}\|)/T, \quad (1)$$

for propagation time $T > 0$. The corresponding forward and backward FTLVs, $\mathbf{l}_i^{\pm}(\pm T, \mathbf{x})$, $i = 1, \ldots, n$, can be computed by choosing the proper vectors from the singular value decomposition of the transition matrix $\Phi(\pm T, \mathbf{x})$ [6], [2]. The orthonormal FTLVs define the following subspaces, for $i = 1, \ldots, n$:

$$\begin{array}{lcl} \mathcal{L}_i^+(T, \mathbf{x}) & := & span\{\mathbf{l}_1^+(T, \mathbf{x}), \ldots, \mathbf{l}_i^+(T, \mathbf{x})\}, \\ \mathcal{L}_i^-(T, \mathbf{x}) & := & span\{\mathbf{l}_i^-(T, \mathbf{x}) \ldots, \mathbf{l}_n^-(T, \mathbf{x})\}, \end{array} \quad (2)$$

Definition [3]: A set $\mathcal{X} \subset \mathbb{R}^n$, $n \geq 2$, is a uniform finite-time two-timescale set for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with fast, slow and convergence time constants (ν^{-1}, σ^{-1} and $\Delta\mu^{-1}$), if there exist positive integers n^s, n^c, n^u, with $n^s + n^c + n^u = n$, $n^c \geq 1$ and $n^s + n^u > 0$, a start time t_s, a cut-off time t_c, and an available averaging time $\overline{T}$ with $0 \leq t_s < t_c \leq \overline{T}$ such that the following three properties are satisfied. We use the notation $\mathcal{T} = (t_s, \overline{T}]$ and $\mathcal{T}_c = (t_s, t_c]$.
1) There exist positive constants $\beta > \alpha > 0$ such that, uniformly on $\mathcal{T} \times \mathcal{X}$, the forward and backward Lyapunov spectra are separated by gaps of size $\Delta\mu = \beta - \alpha$ into n^s, n^c and n^u dimensional subsets.
2) For each $\mathbf{x} \in \mathcal{X}$, there is a continuous splitting of the tangent bundle

$$\begin{array}{l} T_{\mathbf{x}}\mathbb{R}^n = \mathcal{E}^s(\overline{T}, \mathbf{x}) \oplus \mathcal{E}^c(\overline{T}, \mathbf{x}) \oplus \mathcal{E}^u(\overline{T}, \mathbf{x}) \quad \text{where} \\ \mathcal{E}^s = \mathcal{L}_{n^s}^+, \;\; \mathcal{E}^c = \mathcal{L}_{n^s+n^c}^+ \cap \mathcal{L}_{n^s+1}^-, \;\; \mathcal{E}^u = \mathcal{L}_{n^s+n^c+1}^-. \end{array} \quad (3)$$

3) There exist $\nu > \sigma > 0$ such that at each $\mathbf{x} \in \mathcal{X}$ for all $t \in \mathcal{T}_c$

$$\begin{array}{l} \mathbf{v} \in \mathcal{E}^s(\overline{T}, \mathbf{x}) \Rightarrow \begin{cases} \|\Phi(-t, \mathbf{x})\mathbf{v}\| \geq e^{\nu t}\|\mathbf{v}\| \\ \|\Phi(t, \mathbf{x})\mathbf{v}\| \leq e^{-\nu t}\|\mathbf{v}\| \end{cases} \\ \mathbf{v} \in \mathcal{E}^c(\overline{T}, \mathbf{x}) \Rightarrow \;\; e^{-\sigma t}\|\mathbf{v}\| \leq \|\Phi(\pm t, \mathbf{x})\mathbf{v}\| \leq e^{\sigma t}\|\mathbf{v}\| \\ \mathbf{v} \in \mathcal{E}^u(\overline{T}, \mathbf{x}) \Rightarrow \begin{cases} \|\Phi(-t, \mathbf{x})\mathbf{v}\| \leq e^{-\nu t}\|\mathbf{v}\| \\ \|\Phi(t, \mathbf{x})\mathbf{v}\| \geq e^{\nu t}\|\mathbf{v}\| \end{cases} . \end{array} \quad (4)$$

Provided that $\mathcal{X}$ is a uniform finite-time two-timescale set, we can now look for a finite-time n^c-dimensional slow manifold $\mathcal{S}(\overline{T})$ such that $\mathbf{f}(\mathbf{x}) \in \mathcal{E}^c(\overline{T}, \mathbf{x})$ for all $\mathbf{x} \in \mathcal{S}(\overline{T})$. The set

$$\{\mathbf{x} \in \mathcal{X} : \langle \mathbf{f}(\mathbf{x}), \mathbf{w} \rangle = 0, \forall \mathbf{w} \in (\mathcal{E}^c(\overline{T}, \mathbf{x}))^{\perp}\} \quad (5)$$

thus satisfies a necessary condition for a finite-time slow manifold.
Rather than using eigenvectors of $D\mathbf{f}(\mathbf{x})$ as in the ILDM method [1], or direction information from a neighboring manifold [7], we use the appropriate Lyapunov vectors to form the basis for the orthogonal complement of $\mathcal{E}^c(\overline{T}, \mathbf{x})$

$$\begin{array}{r} (\mathcal{E}^c(T, \mathbf{x}))^{\perp} = span\{\mathbf{l}_1^-(T, \mathbf{x}), \ldots, \mathbf{l}_{n^s}^-(T, \mathbf{x}), \\ , \mathbf{l}_{n^s+n^c+1}^+(T, \mathbf{x}), \ldots, \mathbf{l}_n^+(T, \mathbf{x})\}. \end{array} \quad (6)$$

We assume that the manifold can locally be parametrized by a subset of n^c of the n system coordinates and represented as a graph. The n^c independent variables are chosen such that their coordinate axes are not parallel to any directions in $(\mathcal{E}^c)^{\perp}$ and the remaining $n - n^c$ are found by solving the orthogonality conditions in (5).

A. Example - 4D Hamiltonian System

Consider the following Hamiltonian system which arises from the first-order necessary conditions of an optimal control problem

$$\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & -\left(cx_2 + k_1 x_1 + k_2 x_1^3 + \lambda_2/m\right)/m, \\ \dot{\lambda}_1 & = & \lambda_2\left(k_1 + 3k_2 x_1^2\right)/m, \\ \dot{\lambda}_2 & = & -\lambda_1 + c\lambda_2/m. \end{array} \quad (7)$$

We consider (7) in the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with $\mathbf{x} = [x_1\,,\, x_2\,,\, \lambda_1\,,\, \lambda_2]^T \in \mathbb{R}^4$ and $\mathbf{f}$ defined appropriately. For small values of m, the system can be expected to evolve on two disparate timescales. For the numerical results we use $m = 0.5$, $k_1 = 1$, $k_2 = 0.01$, and $c = 4\sqrt{k_1 m}$. FTLA is applied in a region $\mathcal{X}$, chosen such that the ILDM method is applicable yet the slow manifold curvature is large enough that the ILDM method produces noticeable error. We present results for five points that are representative of all the points in $\mathcal{X}$. Figure 1 shows the forward and backward Lyapunov exponents for the five points as functions of the averaging time T. With $n^s = n^u = 1$, $n^c = 2$, $\alpha = 0.5$, $\beta = 5.6$, $\Delta\mu = 5.1$, $\sigma = 0.7$, $\nu = 5.2$, $t_s = 0$ and $t_c = \overline{T} = 0.5$, the conditions, for a uniform two-timescale set resolvable over 2.6 convergence time constants, are satisfied.

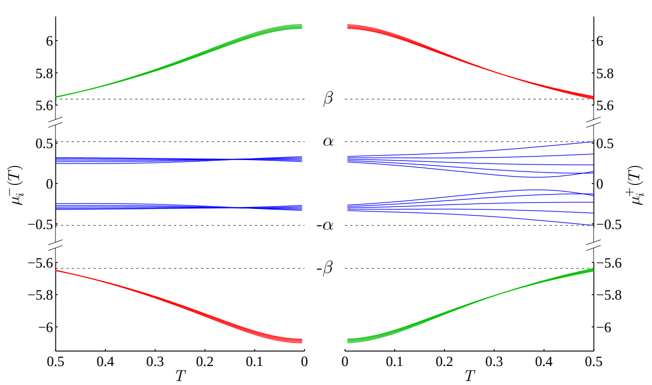

Fig. 1. Superposition of backward and forward FTLEs for points in $\mathcal{X}$. Note that only segments of the y-axis are shown to highlight the central FTLEs.

Computing Slow Manifold Points Using FTLA: The slow subspace $\mathcal{E}^c(\overline{T}, \mathbf{x})$ and its orthogonal complement have dimension $n^c = n - n^c = 2$ and can be written as

$$\begin{array}{rcl} \mathcal{E}^c(\overline{T}, \mathbf{x}) & = & \mathcal{L}_3^+(\overline{T}, \mathbf{x}) \cap \mathcal{L}_2^-(\overline{T}, \mathbf{x}) \\ (\mathcal{E}^c(\overline{T}, \mathbf{x}))^\perp & = & span\{\mathbf{l}_1^-(\overline{T}, \mathbf{x}), \mathbf{l}_4^+(\overline{T}, \mathbf{x})\}. \end{array} \quad (8)$$

We use (x_1, λ_1) as the independent coordinates and compute the (x_2, λ_2) coordinates for the graph of $\mathcal{S}(T)$ by solving the orthogonality conditions in (5). Because the exact location of the slow manifold is not known, we use the following consistency check to assess accuracy. The estimated slow manifold points $\hat{\mathbf{x}}_j$ are propagated backward and forward in time to $\phi(t^\pm, \hat{\mathbf{x}}_j)$. Then for each of the end points, we fix the independent variables, x_1 and λ_1, and use FTLA to recompute the dependent variables, x_2 and λ_2 for the slow manifold point estimate. The degree of consistency between the propagated estimates and re-estimated values of the dependent variables is an indication of accuracy. The same procedure is performed for the ILDM estimates.
Fig. 2, showing points and trajectories projected onto the λ_1-x_2 plane, indicates that FTLA is more consistent than the ILDM method. Although the initial ILDM points (red circles) appear close to the initial FTLA points, the high degree of inconsistency at the end points indicates greater inaccuracy.

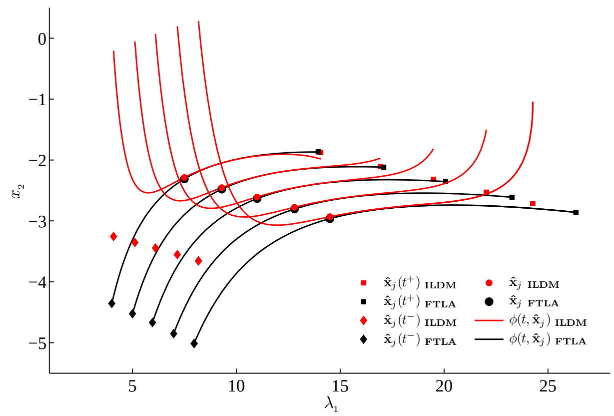

Fig. 2. Projection onto the λ_1-x_2 plane of the forward and backward propagations from initial points on the slow manifold (circles).

References

[1] U. Maas and S. B. Pope, "Simplifying chemical kinetics: intrinsic low-dimensional manifolds in composition space", Combustion and Flame 88 (1992) 239–264.

[2] K. D. Mease, S. Bharadwaj, and S. Iravanchy, "Timescale analysis for nonlinear dynamical systems", J. Guidance, Control and Dynamics 26 (2003) 318-330.

[3] K. D. Mease, U. Topcu, E. Aykutlug, M. Maggia, "Characterizing two-timescale nonlinear dynamics using finite-time Lyapunov exponents and vectors, Characterizing Two-Timescale Nonlinear Dynamics Using Finite-Time Lyapunov Exponents and Vectors", arXiv:0807.0239 [math.DS], 2012. [eprint]

[4] B. Hasselblatt and Y. B. Pesin, "Partially Hyperbolic Dynamical Systems", in B. Hasselblatt and A. Katok (Eds.), Handbook of Dynamical Systems, Vol. 1B, Elsevier, New York, 2005.

[5] A. Adrover, F. Creta, M. Giona, M. Valorani and V. Vitacolonna, "Natural tangent dynamics with recurrent biorthonormalizations: A geometric computational approach to dynamical systems exhibiting slow manifolds and periodic/chaotic limit sets", Physica D 213(2) (2006) 121 – 146.

[6] L. Dieci and E. S. Van Vleck, "Lyapunov spectral intervals: theory and computation", SIAM J. Numerical Analysis 40(2) (2002) 516–542.

[7] B. Rasmussen and L. Dieci, "A geometrical method for the approximation of invariant tori", J. Computational and Applied Mathematics 216 (2008) 388–412.