Characterizing Reduced-Order Manifolds by Finite-Time Lyapunov Analysis

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Personal Trajectory in Model Reduction

- Singular perturbations in flight mechanics

  M. A. Van Buren and K.D. Mease, "*Aerospace Plane Guidance Using Time-Scale Decomposition and Feedback Linearization*", J. Guidance Control, and Dynamics, Vol. 15, No. 5, pp. 1166-1174,1992.

- Fenichel geometric singular perturbations

  M. A. Van Buren and K.D. Mease, "*Geometric Synthesis of Aerospace Plane Ascent Guidance*", Automatica , Vol. 30, No. 12, pp. 1839-1849, 1994.

- Discussions with S.-H. Lam: ILDM and CSP

  K.D. Mease, "*Geometry of Computational Singular Perturbations In Nonlinear Control System Design*", ed. A. J. Krener and D. Q. Mayne, Oxford, UK: Pergamon, pp. 855-861, Vol. 2, 1996.

- Differential geometry and dynamical systems: Lai-Sang Young, Yakov Pesin, et al.

- Finite-time Lyapunov exponents and vectors for timescale analysis

  K. D. Mease, S. Bharadwaj, and S. Iravanchy, "*Timescale Analysis for Nonlinear Dynamical Systems, J. Guidance, Control and Dynamics*", Vol. 26, No. 2, pp. 318-330, 2003.

  K.D. Mease, "*Multiple Time-Scales in Nonlinear Flight Mechanics: Diagnosis and Modeling*", Applied Mathematics and Computation, Vol. 164, pp. 627-648, 2005.

Nonlinear System and Tangent Linear Dynamics

- Nonlinear ODE System

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \longrightarrow \quad \mathbf{x}(t) = \phi(t, \mathbf{x}_0) \qquad \mathbf{x} \in \mathbb{R}^n$$

- Nonlinear ODE System & Tangent Linear Dynamics $(\mathrm{Df(x)}{=}\partial\mathbf{f}(\mathbf{x})/\partial\mathbf{x})$

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \\ \dot{\mathbf{v}} = D\mathbf{f}(\mathbf{x})\ \mathbf{v} \\ \mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{v}(0) = \mathbf{v}_0 \end{cases} \longrightarrow \begin{cases} \mathbf{x}(t) = \phi(t, \mathbf{x}_0) \\ \mathbf{v}(t) = \Phi(t, \mathbf{x}_0)\mathbf{v_0} \end{cases} \quad (\mathbf{x}, \mathbf{v}) \in T\mathbb{R}^n$$

State Transition Matrix:

$$\begin{cases} \dot{\Phi} = D\mathbf{f}(\mathbf{x})\ \Phi \\ \Phi(0, \mathbf{x}_0) = I_{n\times n} \end{cases}$$

Slow Manifold Characterization Methods

- Methods for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
  - Roussel-Fraser
  - Quasi Steady-State Approximation (*QSSA*)
  - Zero-Derivative Principle (*ZDP*)
  - Singular Perturbations

- Methods for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \;\; \& \;\; \dot{\mathbf{v}} = D\mathbf{f}(\mathbf{x})\,\mathbf{v}$
  - Intrinsic Low-Dimensional Manifold (*ILDM*):
    - $eig(D\mathbf{f}(\mathbf{x})) + \text{orthogonality conditions}$
  - Computational Singular Perturbations (*CSP*):
    - $eig(D\mathbf{f}(\mathbf{x})) + \text{invariance} + \text{simulation}$
  - Finite-Time Lyapunov Analysis (*FTLA*):
    - $eig(\Phi^T\Phi)\ ,\ eig(\Phi\Phi^T) + \text{orthogonality conditions}$

Geometry of Two Timescale (2D System)

2D LTI System

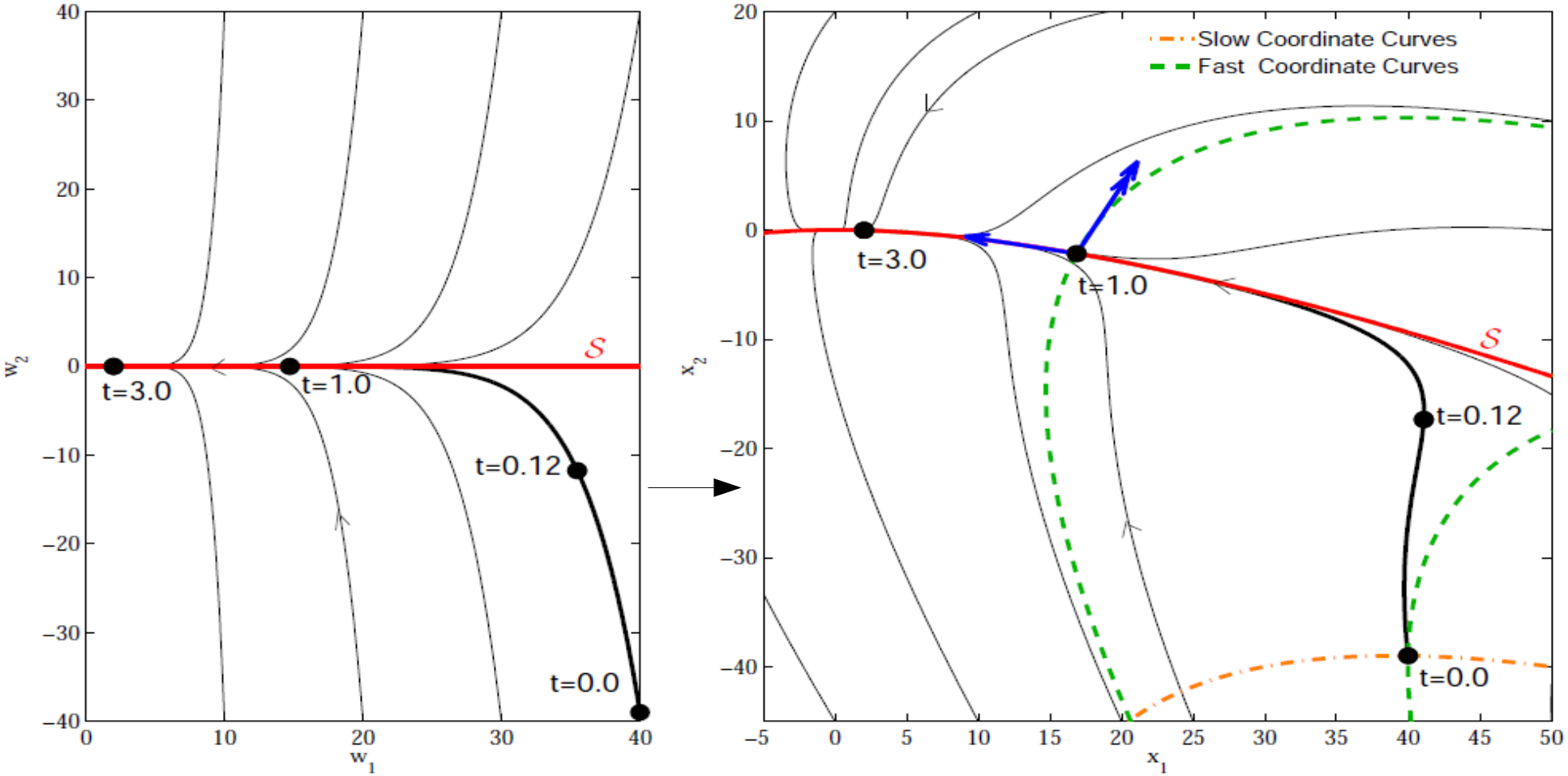
$$\begin{cases} \dot{w}_1 = -w_1 \\ \dot{w}_2 = -10 w_2 \end{cases}$$

Nonlinear Transformation

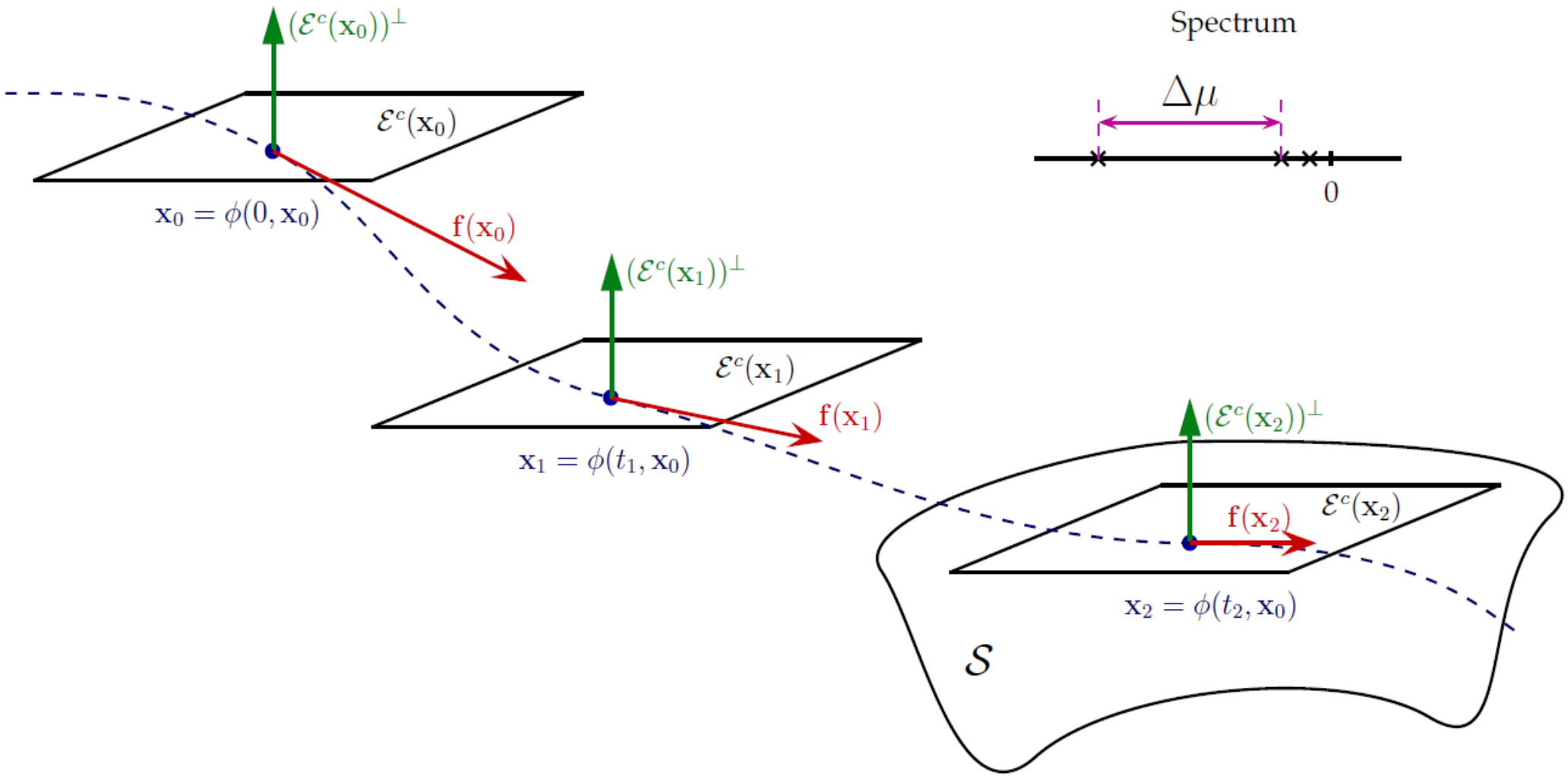
$$w_1 = x_1 - a^2(x_1 + x_2)^2$$
$$w_2 = x_2 + a^2(x_1 + x_2)^2$$

2D NTI System

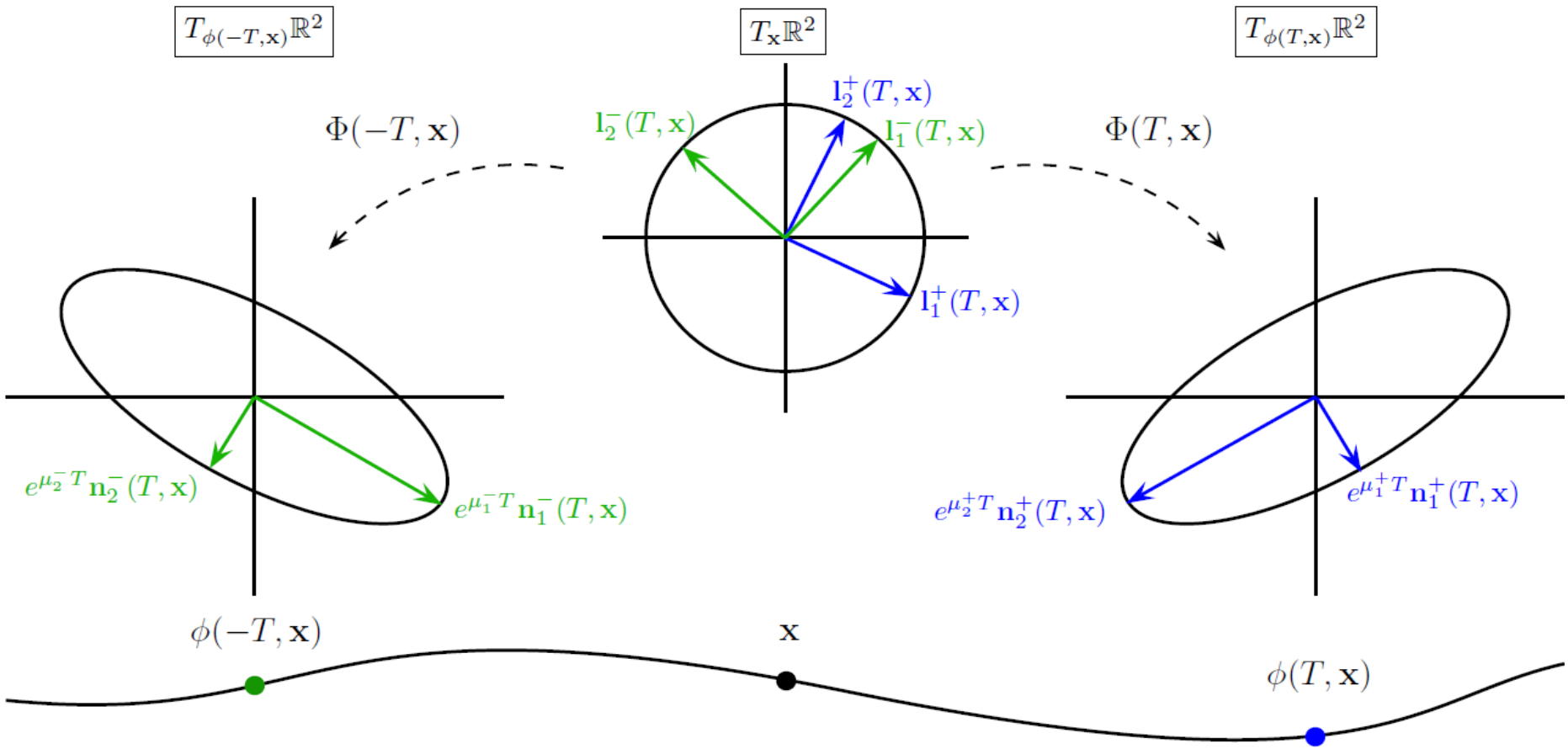
$$\begin{cases} \dot{x}_1 = -x_1 - a(x_1 + 19x_2)(x_1 + x_2) - 18a^2(x_1 + x_2)^3 \\ \dot{x}_2 = -10x_2 - 2a(4x_1 - 5x_2)(x_1 + x_2) + 18a^2(x_1 + x_2)^3 \end{cases}$$



Geometry of Two Timescale (3D System)



Finite-Time Lyapunov Analysis



SVD $\Phi(\pm T,\mathbf{x}) = N^{\pm}(T,\mathbf{x})\Sigma^{\pm}(T,\mathbf{x})L^{\pm}(T,\mathbf{x})^T$

FTLVs: $\mathbf{l}_i^{\pm}(\pm T,\mathbf{x})$ columns of $L^{\pm}(T,\mathbf{x})$

FTLEs: $\mu_i^{\pm}(T,\mathbf{x},\mathbf{l}_i^{\pm}) := \frac{1}{T}\ln\|\Phi(\pm T,\mathbf{x})\mathbf{l}_i^{\pm}\|/\|\mathbf{l}_i^{\pm}\|$

$$\mathcal{L}_i^+(T,\mathbf{x}) = span\{l_1^+(T,\mathbf{x}),...,l_i^+(T,\mathbf{x})\}$$

$$\mathcal{L}_i^-(T,\mathbf{x}) = span\{l_i^-(T,\mathbf{x}),...,l_n^-(T,\mathbf{x})\}$$

Finite-Time Two Timescale Set – Definition

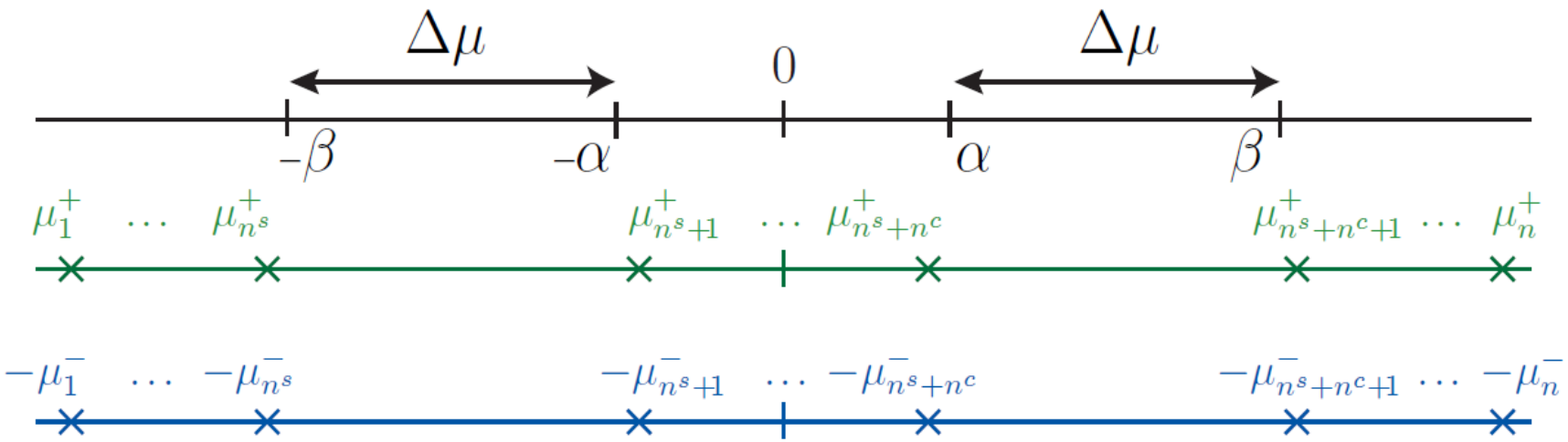
Definition. A set $\mathcal{X} \subset \mathbb{R}^n$, $n \geq 2$, is a uniform finite-time two-timescale set for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with fast, slow and convergence time constants ν^{-1}, σ^{-1}, $\Delta\mu^{-1}$, resolvable over $\Delta\mu(\overline{T} - t_s)$ convergence time constants, if there exist positive integers n^s, n^c and n^u, with $n^s + n^c + n^u = n$, a start time t_s, a cut-off time t_c, and an available averaging time $\overline{T}$ with $0 \leq t_s < t_c \leq \overline{T}$ such that the following three properties are satisfied. We use the notation $\mathcal{T} = (t_s, \overline{T}]$ and $\mathcal{T}_c = (t_s, t_c]$.

1. Uniform Spectral Gaps
2. Tangent Bundle Splitting
3. Two Timescales (Exponential Bounds)

1) Uniform Spectral Gaps

There exist $\beta > \alpha > 0$ such that, uniformly on $\mathcal{T} \times \mathcal{X}$, the forward and backward Lyapunov spectra are separated by gaps of size $\Delta\mu = \beta - \alpha$ into n^s, n^c and n^u dimensional subsets specified by:

$$\mu^+_{n^s} \leq -\beta, \quad -\alpha \leq \mu^+_{n^s+1}, \quad \mu^+_{n^s+n^c} \leq \alpha, \quad \beta \leq \mu^+_{n^s+n^c+1},$$
$$-\mu^-_{n^s} \leq -\beta, \quad -\alpha \leq -\mu^-_{n^s+1}, \quad -\mu^-_{n^s+n^c} \leq \alpha, \quad \beta \leq -\mu^-_{n^s+n^c+1}.$$



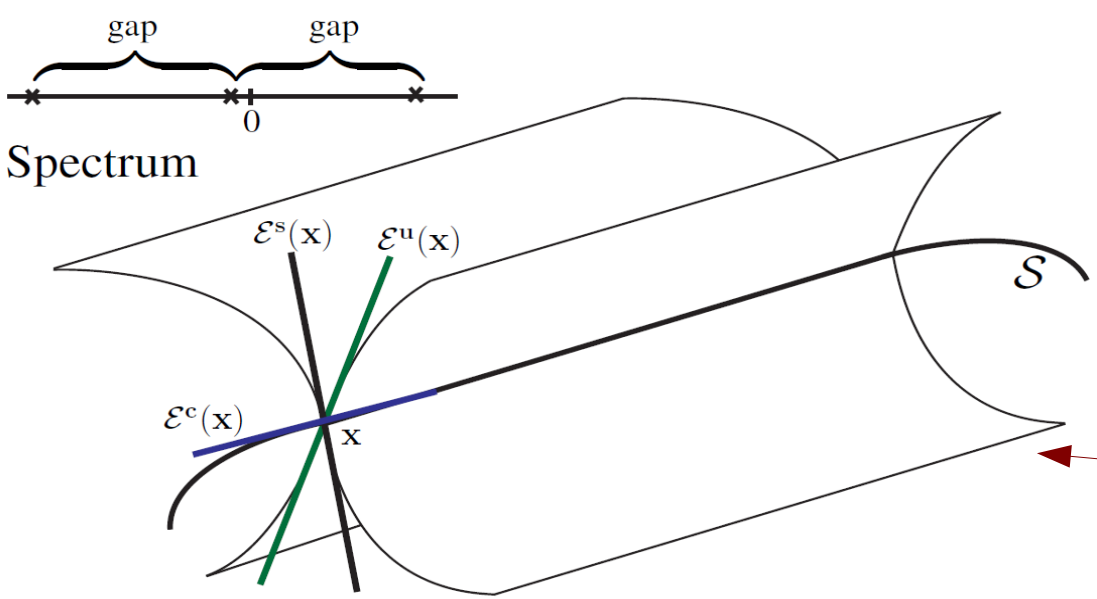
2) Tangent Bundle Splitting

For each $\mathbf{x} \in \mathcal{X}$, there is a continuous splitting

$$T_{\mathbf{x}}\mathbb{R}^n = \mathcal{E}^s(\overline{T}, \mathbf{x}) \oplus \mathcal{E}^c(\overline{T}, \mathbf{x}) \oplus \mathcal{E}^u(\overline{T}, \mathbf{x}),$$

where

$$\begin{aligned}
\mathcal{E}^s(\overline{T}, \mathbf{x}) &= \mathcal{L}^+_{n^s}(\overline{T}, \mathbf{x}), \\
\mathcal{E}^c(\overline{T}, \mathbf{x}) &= \mathcal{L}^+_{n^s+n^c}(\overline{T}, \mathbf{x}) \cap \mathcal{L}^-_{n^s+1}(\overline{T}, \mathbf{x}), \\
\mathcal{E}^u(\overline{T}, \mathbf{x}) &= \mathcal{L}^-_{n^s+n^c+1}(\overline{T}, \mathbf{x}).
\end{aligned}$$



3D system with 1D normally hyperbolic slow manifold. $(n^s = n^c = n^u = 1)$

3) Two-Timescale Set (Exponential Bounds)

There exist positive numbers ν and σ with $\nu > \sigma$ such that at each $\mathbf{x} \in \mathcal{X}$ for all $t \in \mathcal{T}_c$

$$\mathbf{v} \in \mathcal{E}^s(\overline{T}, \mathbf{x}) \Rightarrow \begin{cases} \|\Phi(-t, \mathbf{x})\mathbf{v}\| \geq e^{\nu t}\|\mathbf{v}\| \\ \|\Phi(t, \mathbf{x})\mathbf{v}\| \leq e^{-\nu t}\|\mathbf{v}\| \end{cases},$$

$$\mathbf{v} \in \mathcal{E}^c(\overline{T}, \mathbf{x}) \Rightarrow \begin{cases} e^{-\sigma t}\|\mathbf{v}\| \leq \|\Phi(t, \mathbf{x})\mathbf{v}\| \leq e^{\sigma t}\|\mathbf{v}\| \\ e^{-\sigma t}\|\mathbf{v}\| \leq \|\Phi(-t, \mathbf{x})\mathbf{v}\| \leq e^{\sigma t}\|\mathbf{v}\| \end{cases},$$

$$\mathbf{v} \in \mathcal{E}^u(\overline{T}, \mathbf{x}) \Rightarrow \begin{cases} \|\Phi(-t, \mathbf{x})\mathbf{v}\| \leq e^{-\nu t}\|\mathbf{v}\| \\ \|\Phi(t, \mathbf{x})\mathbf{v}\| \geq e^{\nu t}\|\mathbf{v}\| \end{cases}.$$

Slow Manifold Characterization Procedure

1) Determine *FTLEs* spectrum on a grid of points, check if definition satisfied.

2) Use *FTLVs* to formulate orthogonality conditions.

- A finite-time Slow Manifold is a n^c-dimensional manifold such that:

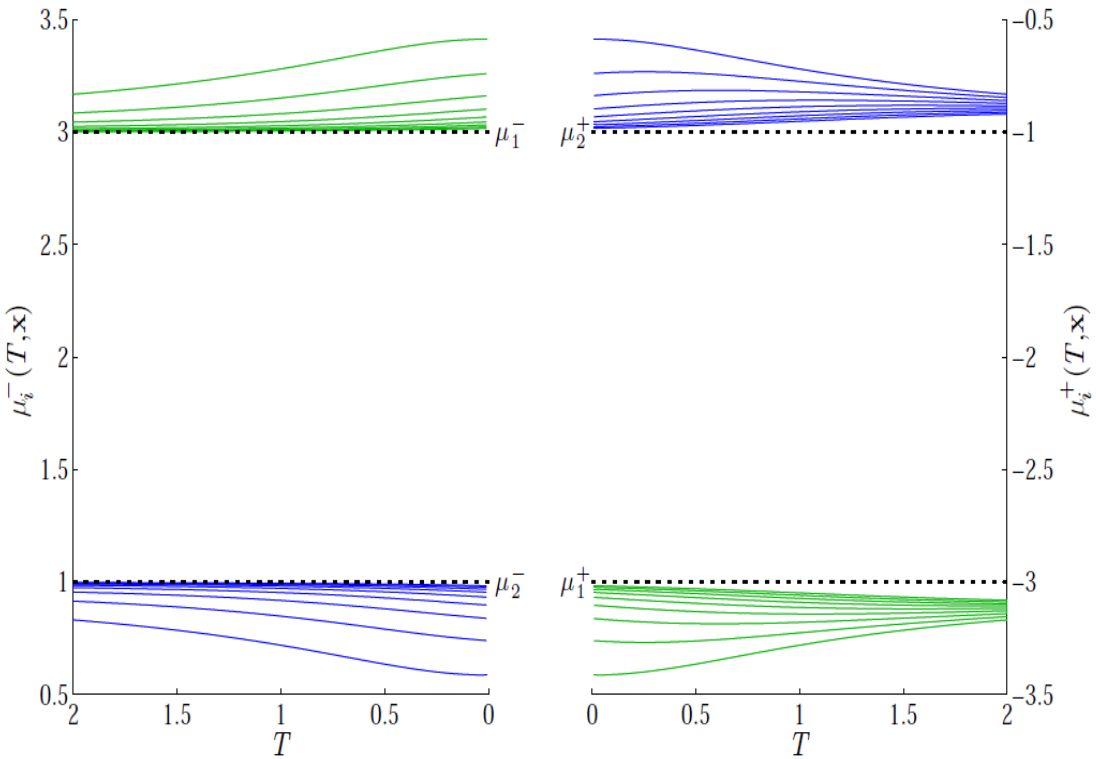
$$\forall \mathbf{x} \in \mathcal{S}(T) \ : \ T_x\mathcal{S} = \mathcal{E}^c(T, \mathbf{x}) \ , \ \mathbf{f}(\mathbf{x}) \in \mathcal{E}^c(T, \mathbf{x}).$$

- The following set of points satisfies necessary condition for a finite-time slow manifold:

$$\{\mathbf{x} \in \mathcal{X} : \langle \mathbf{f}(\mathbf{x}), \mathbf{w} \rangle = 0, \forall \mathbf{w} \in [\mathcal{E}^c(T, \mathbf{x})]^{\perp}\}$$

3) Compute slow manifold points.

Example I: 2D Nonlinear System (Davis-Skodje)



2D Nonlinear system by Davis-Skodje

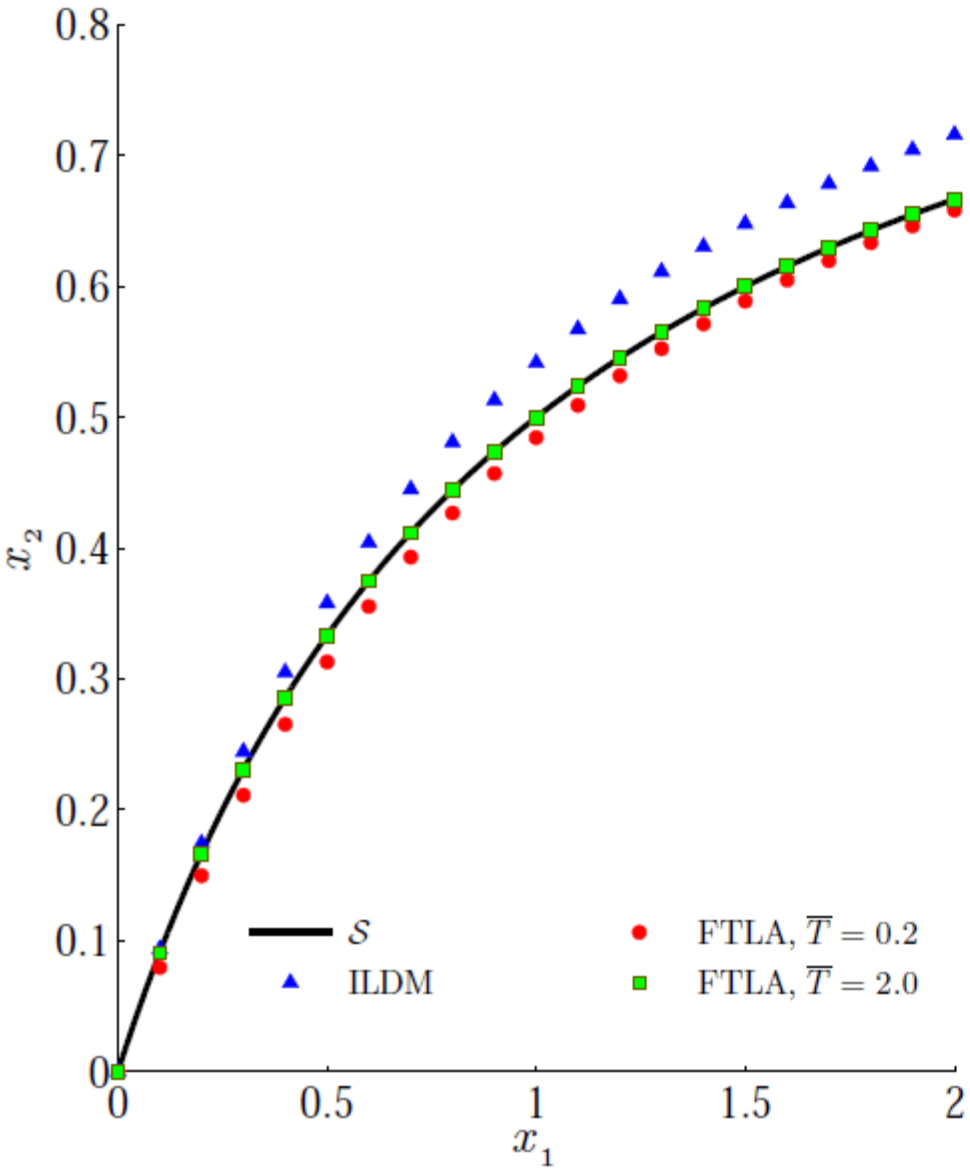
$$\dot{x}_1 = -x_1,$$
$$\dot{x}_2 = -\gamma x_2 + \frac{(\gamma-1)x_1+\gamma x_1^2}{(1+x_1)^2}$$

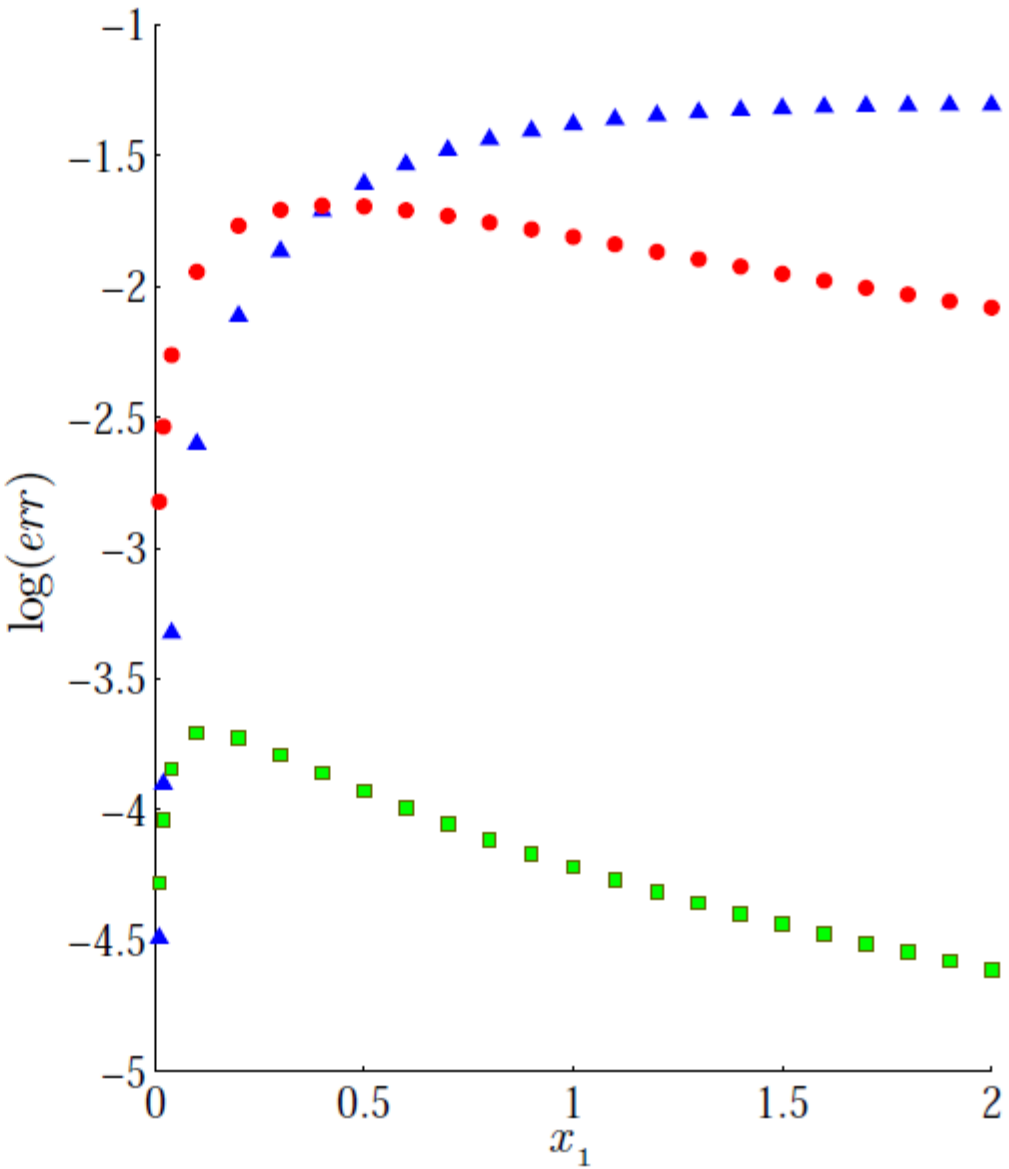
$$\mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \ , \ x_2 \geq 0\}$$

For $\gamma = 3$, trajectories are attracted to the 1D slow manifold

$$\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1/(1 + x_1)\}$$

Example I: Points on Slow Manifold (FTLA vs ILDM)





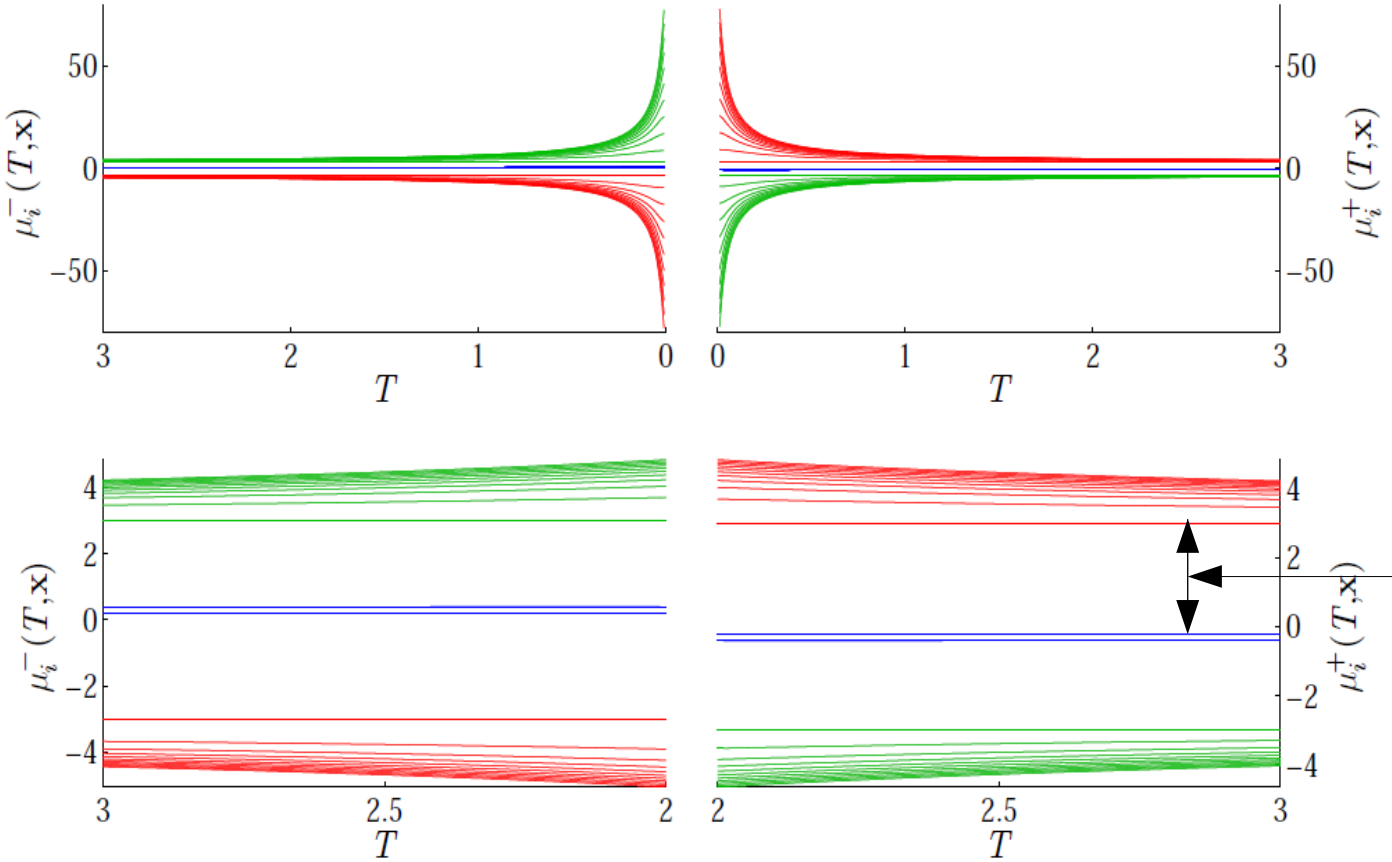
Example II: 3D Nonlinear System

Consider the following 3D Nonlinear System

$$\begin{aligned} \dot{x}_1 &= ax_1 \\ \dot{x}_2 &= bx_2 + \gamma(b-2a)x_1^2 \\ \dot{x}_3 &= cx_3 + \gamma(c-2a)x_1^2 \end{aligned}$$

with $a = -0.2, \gamma = 2$, $b = -3, c = 3,$

$$n^s = n^c = n^u = 1$$

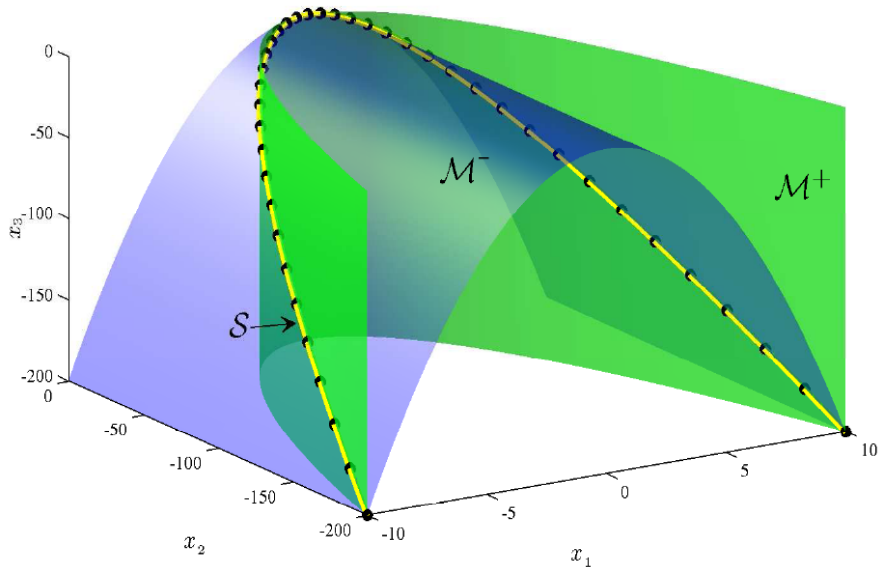


The FTLEs are computed on a grid on the cubic region

$$\mathcal{X} = [-10, 10] \subset \mathbb{R}^3$$

$$\Delta\mu = 2.2$$

Example II: 3D Nonlinear System (Slow Manifold)



$$\mathcal{M}^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 + \gamma x_1^2 = 0\}$$
$$\mathcal{M}^- = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 + \gamma x_1^2 = 0\}$$

Exact Slow Manifold: $\mathcal{S} = \mathcal{M}^+ \bigcap \mathcal{M}^-$

ILDM

$$\langle \mathbf{w}_1, \mathbf{f}(\mathbf{x}) \rangle = 0, \quad \langle \mathbf{w}_2, \mathbf{f}(\mathbf{x}) \rangle = 0$$

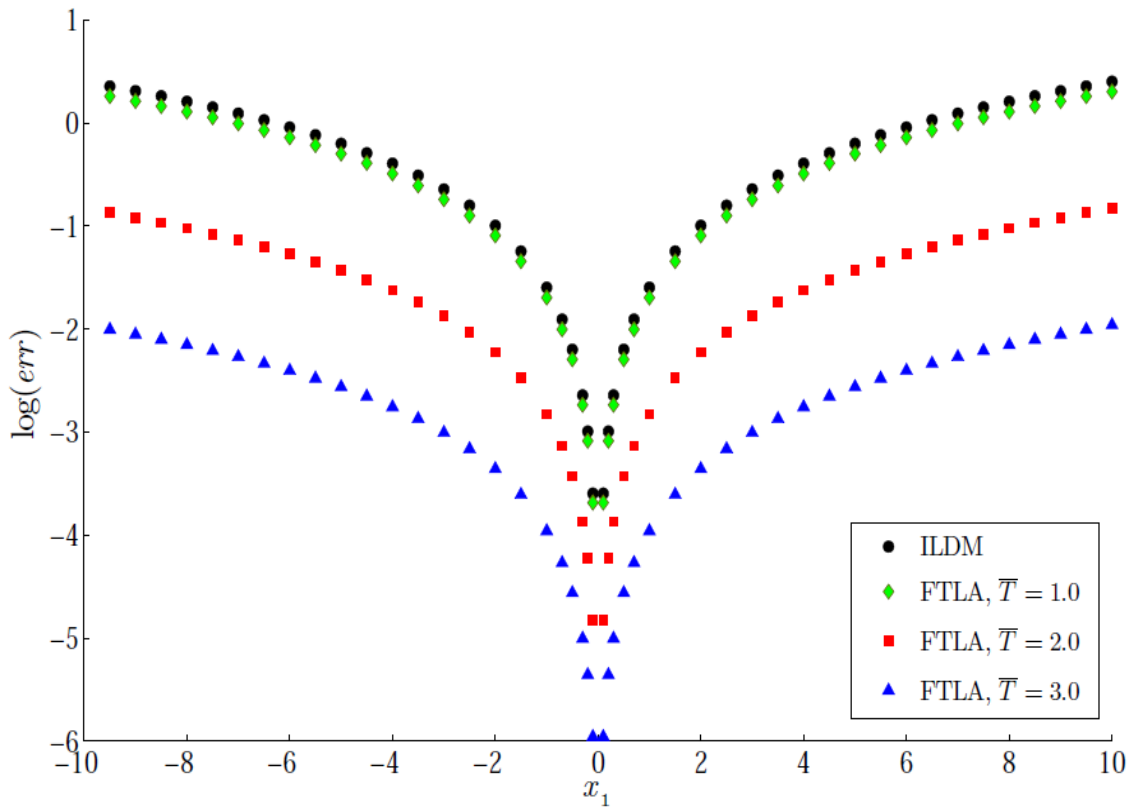
$$\mathbf{w}_1 = \left[2\gamma x_1 \left(\frac{b-2a}{b-a} \right), \quad 1, \quad 0 \right]^T$$

$$\mathbf{w}_2 = \left[2\gamma x_1 \left(\frac{c-2a}{c-a} \right), \quad 0, \quad 1 \right]^T$$

FTLA

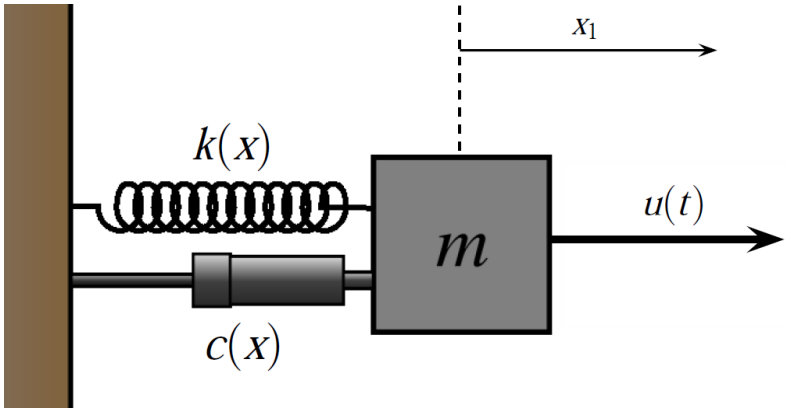
$$\langle \mathbf{f}(\mathbf{x}), \mathbf{w} \rangle = 0, \mathbf{w} \in [\mathcal{E}^c(\overline{T}, \mathbf{x})]^{\perp}\}$$

$$(\mathcal{E}^c(\overline{T}, \mathbf{x}))^{\perp} = span\{\mathbf{l}_1^-(\overline{T}, \mathbf{x}), \mathbf{l}_3^+(\overline{T}, \mathbf{x})\}$$



Example III: 4D Hamiltonian System

- Consider the Optimal Control Problem modeled as:



$$min \quad J = \int_0^{t_f} \frac{1}{2} u^2 \, \mathrm{d}t \quad \text{subject to:}$$

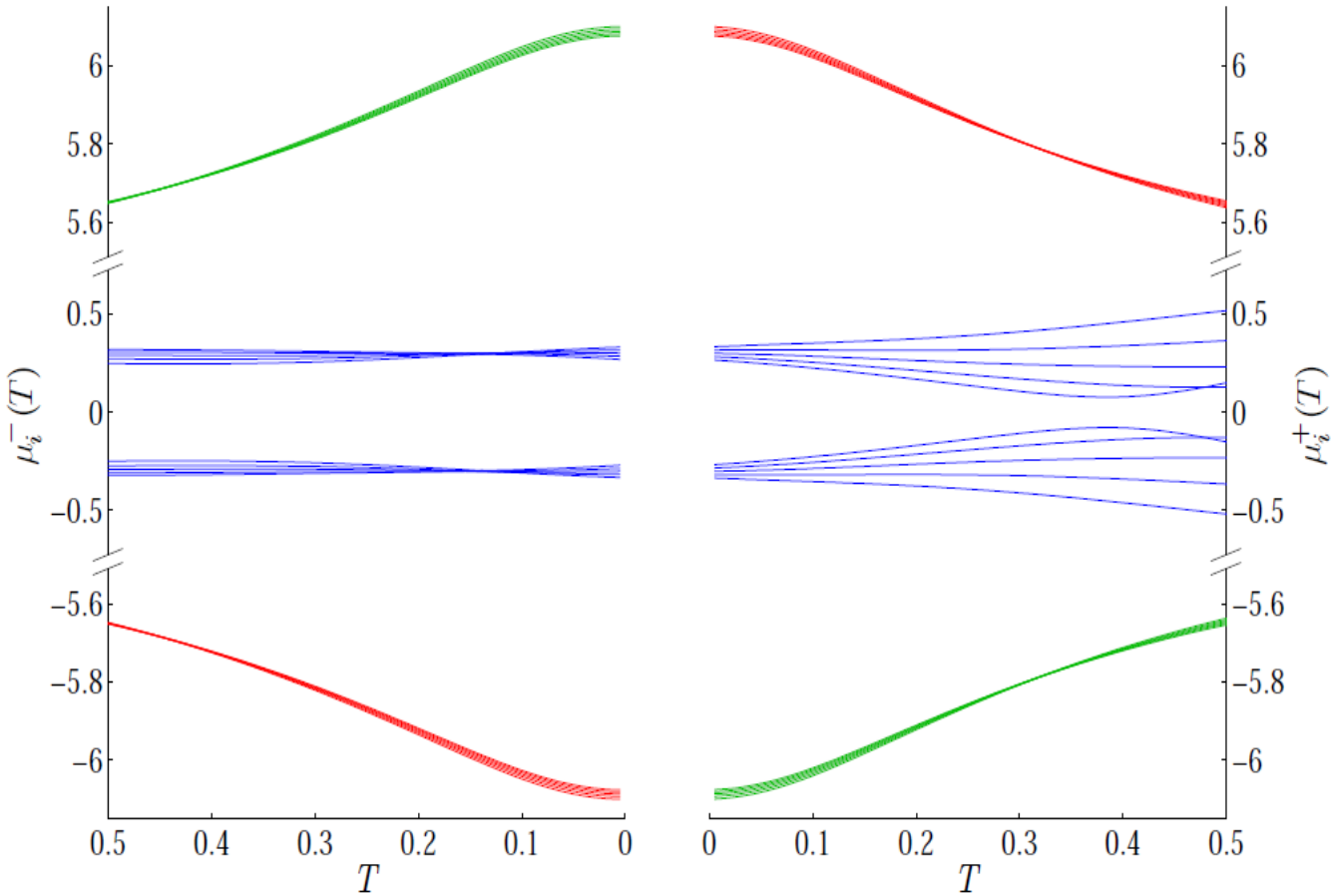
$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\tfrac{1}{m}\left(c x_2 + k_1 x_1 + k_2 x_1^3\right) + \tfrac{u}{m}. \end{aligned}$$

- The First-Order Necessary Conditions lead to the Hamiltonian system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\tfrac{1}{m}\left(c x_2 + k_1 x_1 + k_2 x_1^3 + \tfrac{\lambda_2}{m}\right), \\ \dot{\lambda}_1 &= \tfrac{\lambda_2}{m}\left(k_1 + 3 k_2 x_1^2\right), \\ \dot{\lambda}_2 &= -\lambda_1 + c\tfrac{\lambda_2}{m}. \end{aligned}$$

Example III: diagnosing two timescales

- Using $\mathrm{m}{=}0.5\,,\,\mathrm{k}_1 = 1\,,\; k_2 = 0.01\,,\; c = 4\sqrt{k_1 m}$ and considering points in a region $\mathcal{X}$ such that ILDM method can be applied:



- The system has two timescales with a gap $\Delta\mu = 5.1$ for $\overline{T} = 0.5$
- The convergence of the FTLVs is guaranteed over *2.6* convergence time constants.
- $n^s = n^u = 1, n^c = 2$

Example III: slow subspace and orthogonality conditions

- The Slow subspace $\mathcal{E}^c(\overline{T}, \mathbf{x})$ has dimension 2 and can be written as:

$$\mathcal{E}^c(\overline{T}, \mathbf{x}) \quad = \quad \mathcal{L}_3^+(\overline{T}, \mathbf{x}) \cap \mathcal{L}_2^-(\overline{T}, \mathbf{x})$$

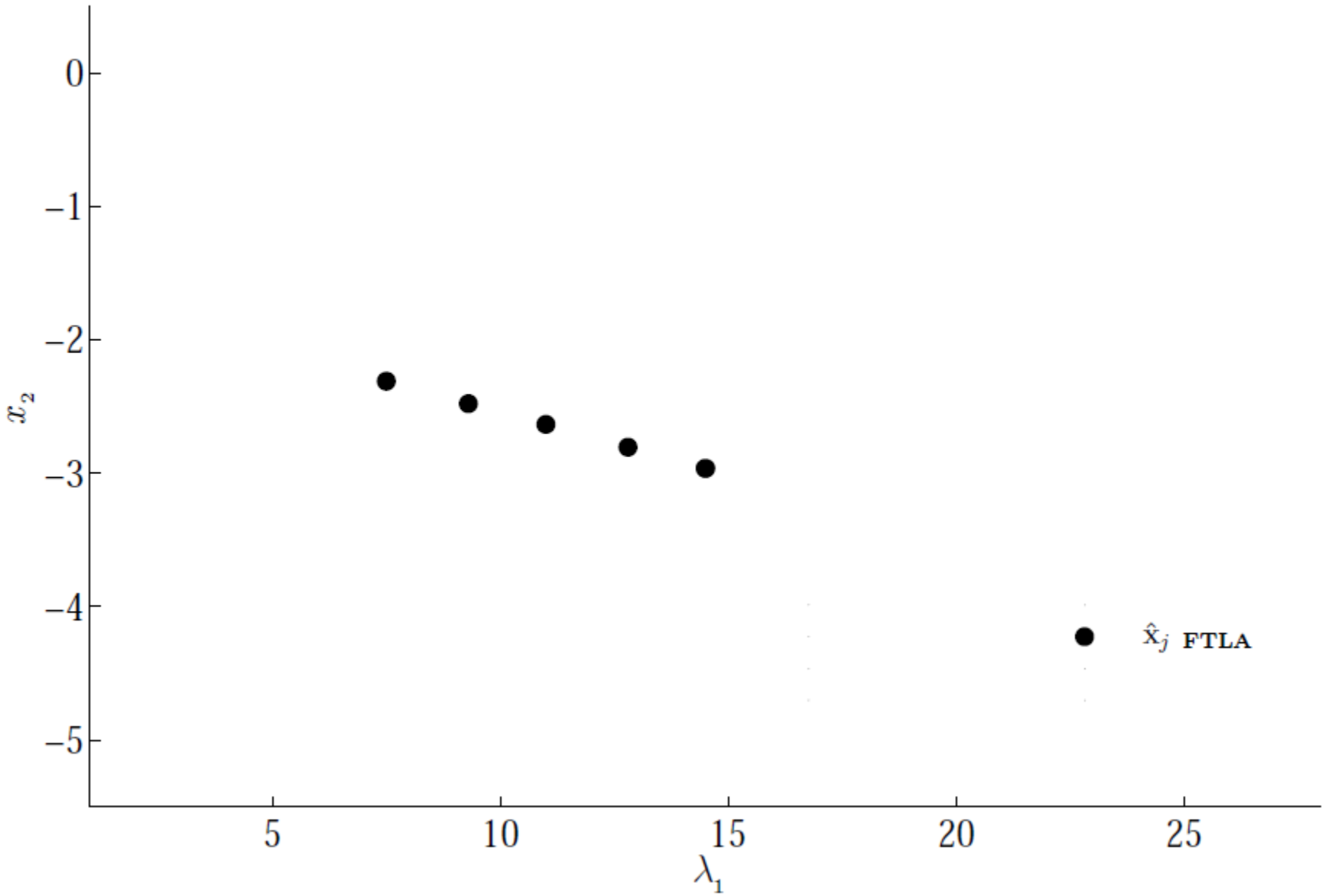
With its orthogonal complement:

$$(\mathcal{E}^c(\overline{T}, \mathbf{x}))^\perp = span\{\mathbf{l}_1^-(\overline{T}, \mathbf{x}), \mathbf{l}_4^+(\overline{T}, \mathbf{x})\}.$$

- Fixing $n^c = 2$ independent variables, the remaining $n - n^c = 2$ are found by solving the following orthogonality conditions:

$$\langle \mathbf{l}_1^-(\overline{T}, \mathbf{x}), \mathbf{f}(\mathbf{x}) \rangle = 0$$
$$\langle \mathbf{l}_4^+(\overline{T}, \mathbf{x}), \mathbf{f}(\mathbf{x}) \rangle = 0$$

Example III: points on Slow Manifold



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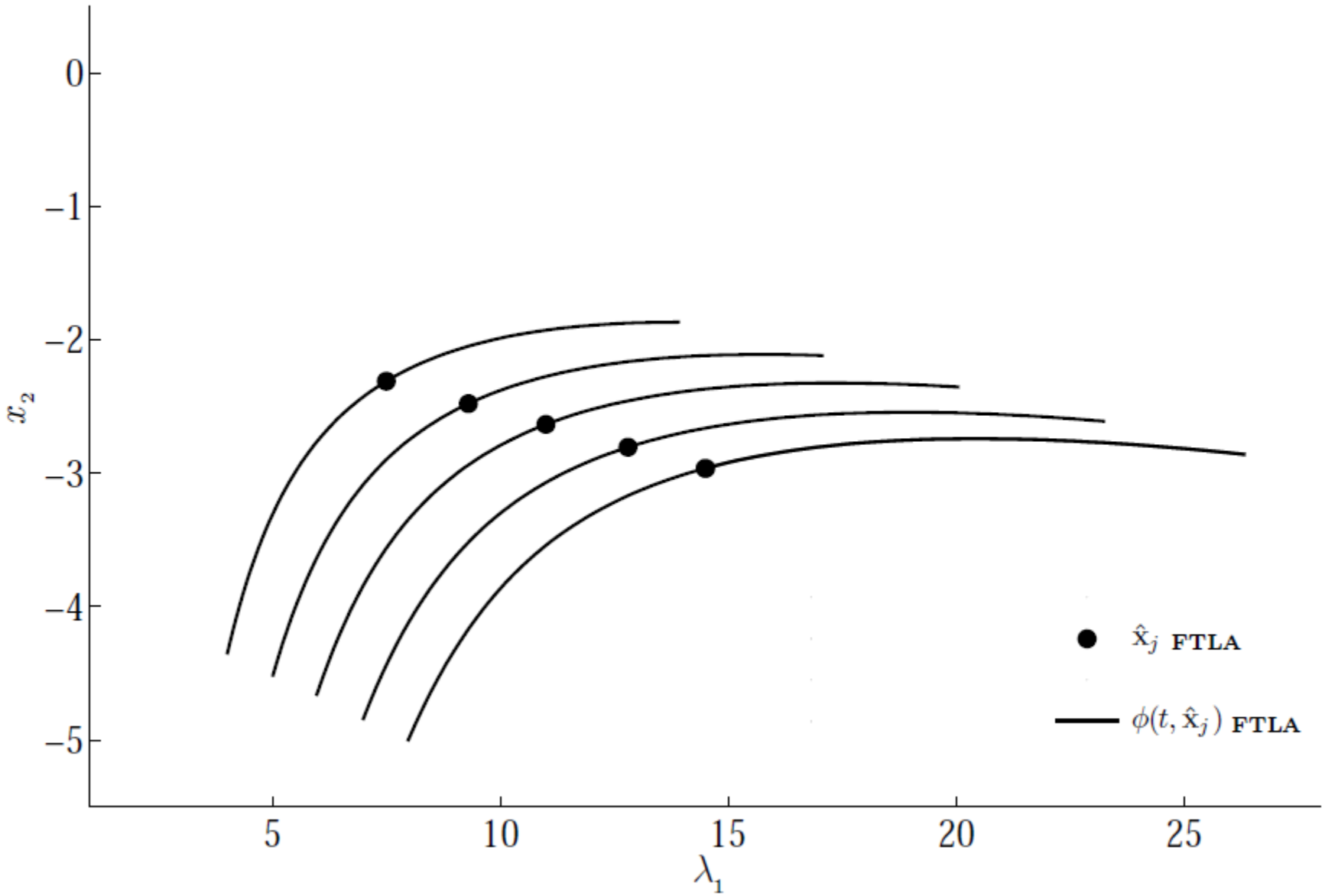

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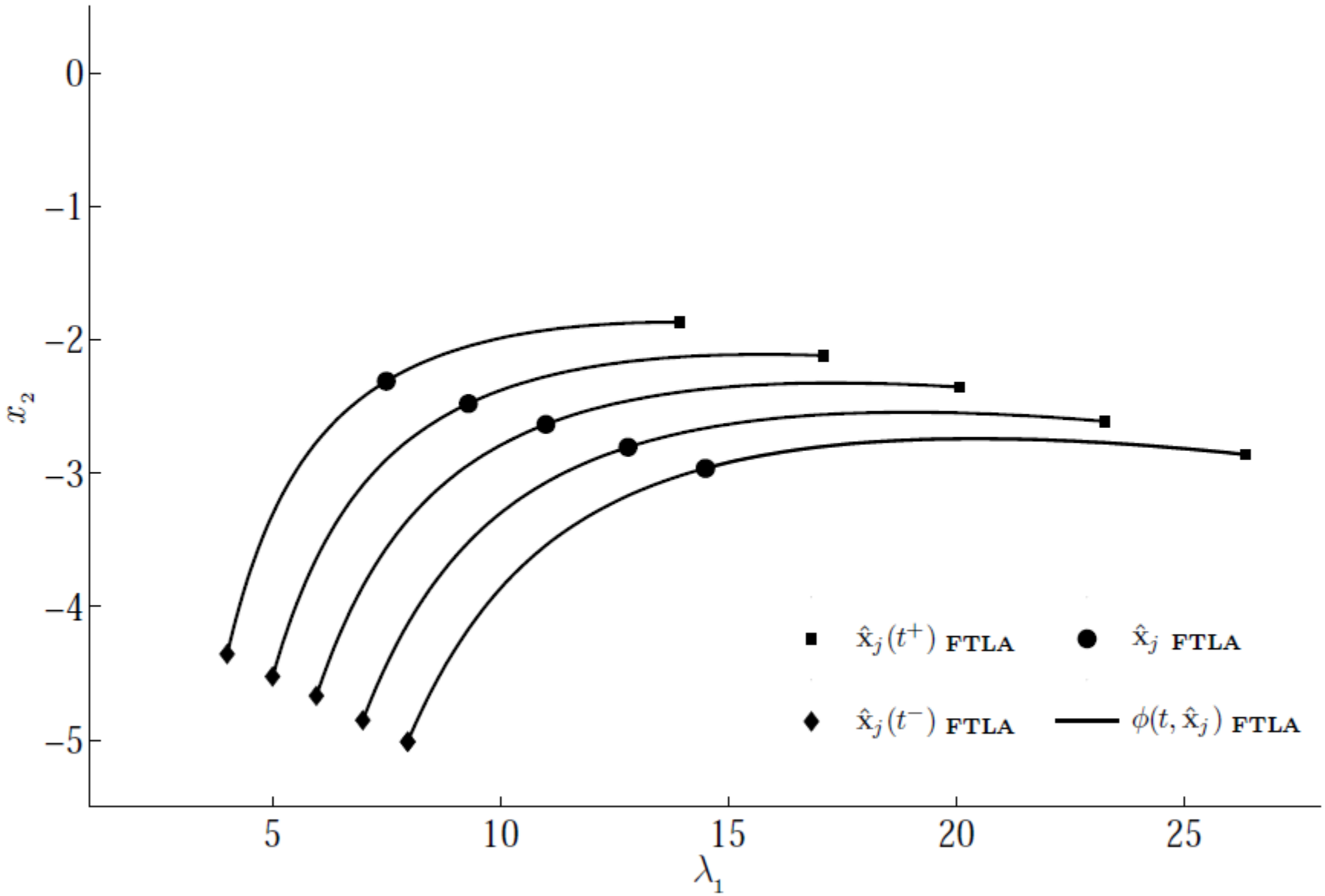

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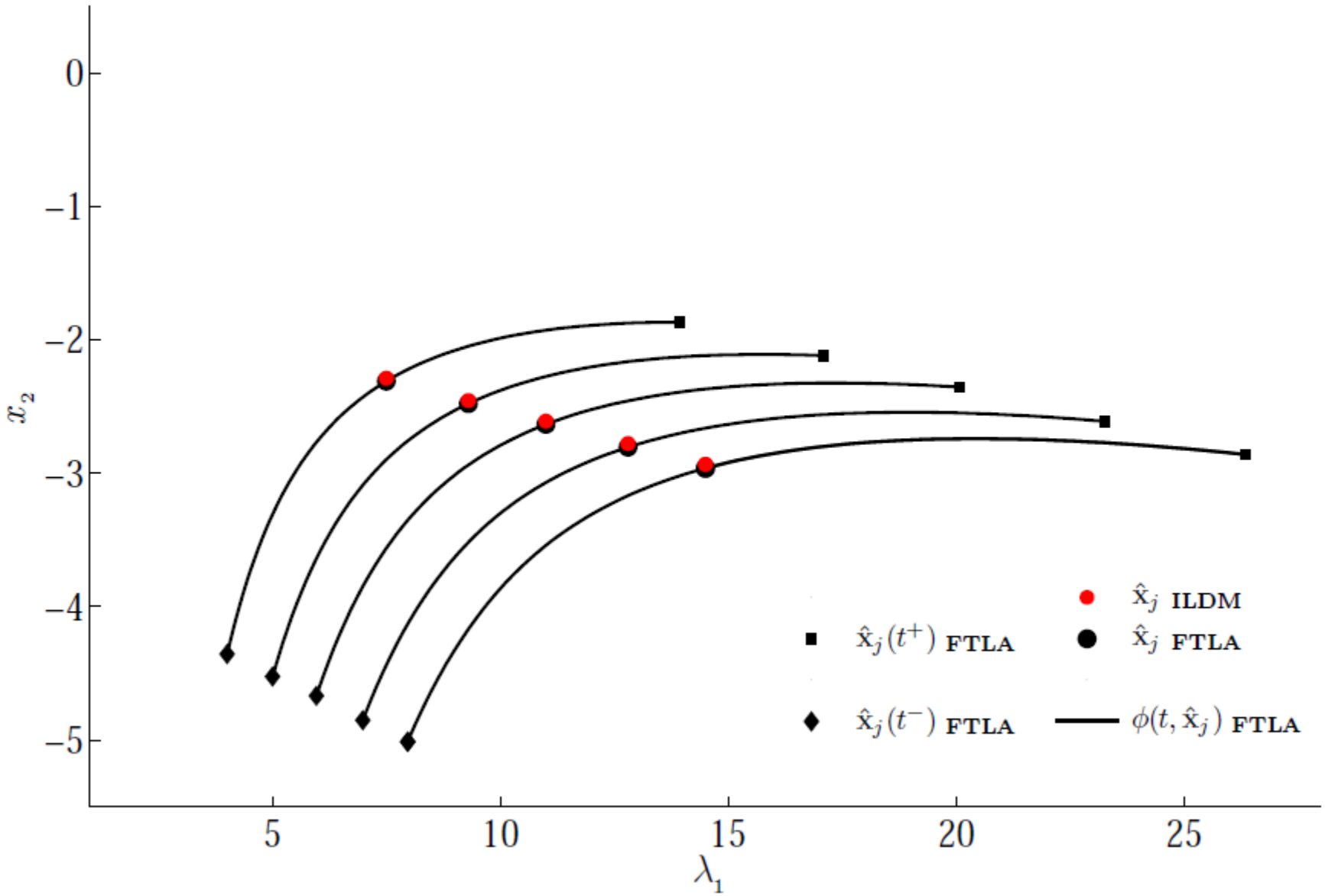

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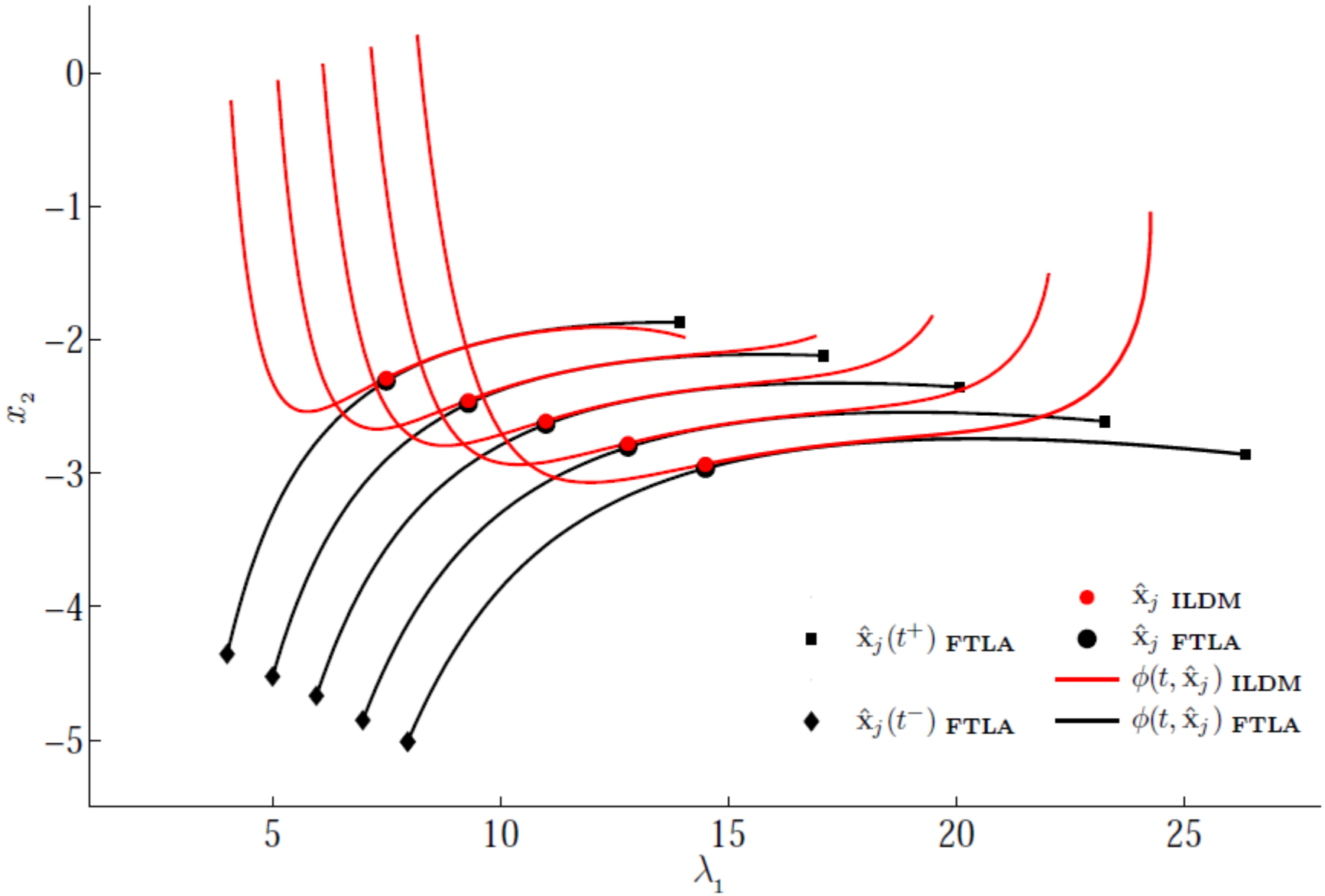
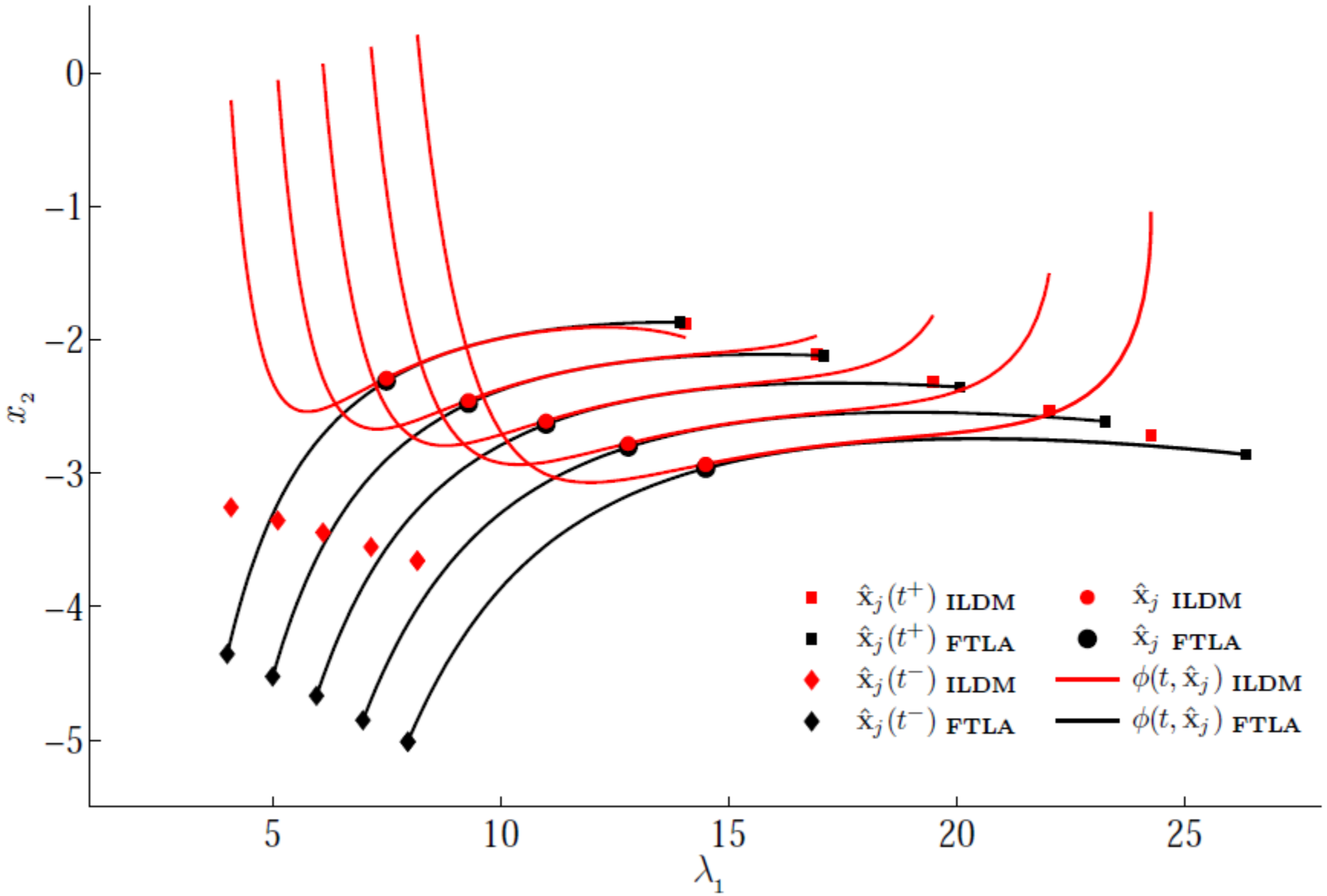
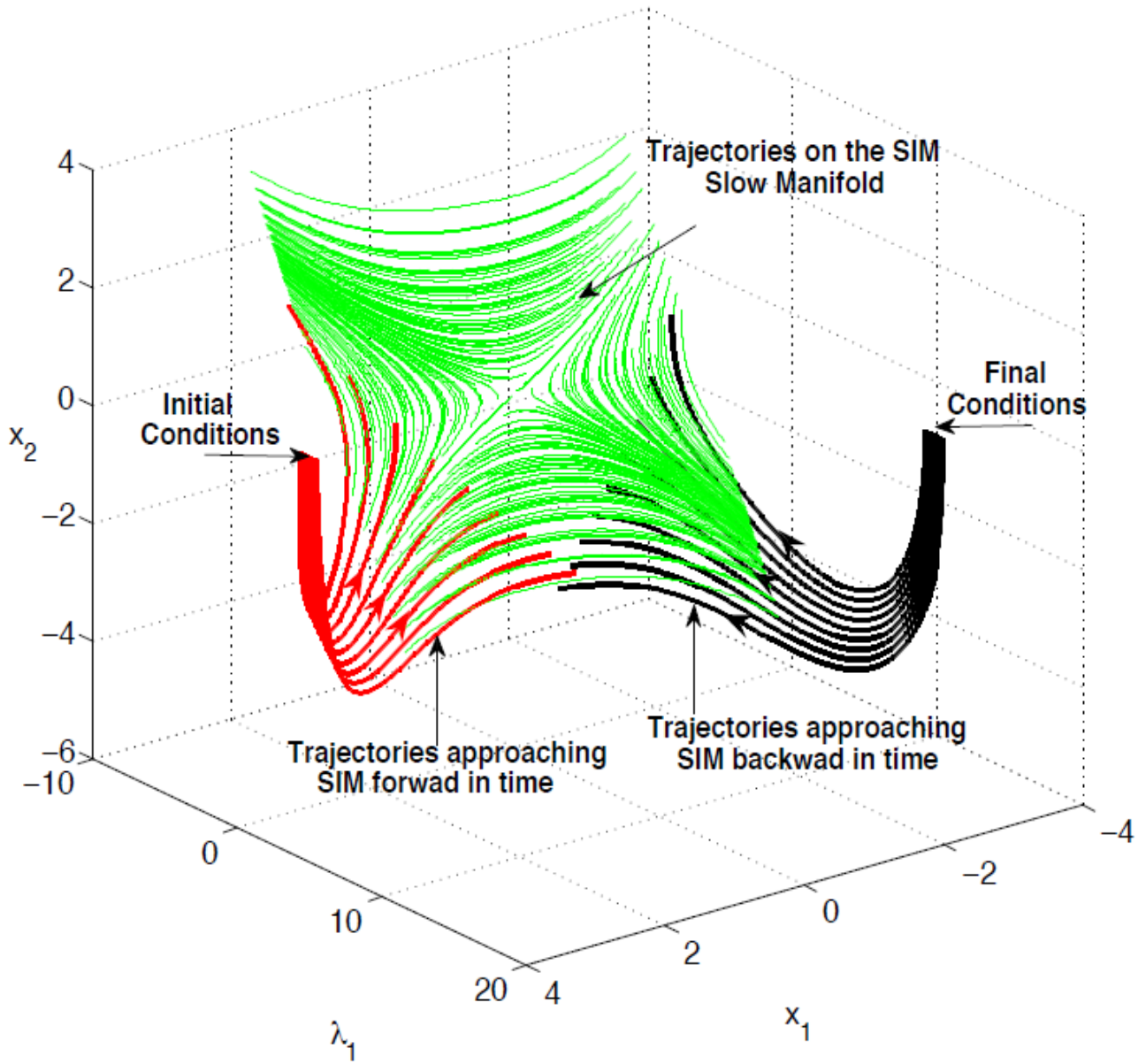


Example III: points on Slow Manifold



Example III: matching trajectories on Slow Manifold



- Trajectories departing from initial and final conditions for the Hamiltonian BVP
- The trajectories match on the Slow Manifold

Conclusion and Current Directions

- Conclusion:
  - FTLA method of characterizing slow manifold most closely related to ILDM, requires more computation, greater accuracy, consistent with theory of partially hyperbolic sets in asymptotic limit of averaging time. Demonstrated for low dimensional systems.
- Current directions:
  - Approximate solution of Hamiltonian BVPs from Optimal Control, using manifold structure.
  - Model reduction for climate simulation.
- Connections:
  - Weather predictability.
  - Lagrangian coherent structures in fluids.
  - Orbital stability and manifold structure in astrodynamics.