Asymptotic behavior of a reaction-diffusion equation perturbed by multiplicative noise

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***Abstract*— We study the asymptotic behaviour of a reaction-diffusion equation, and prove that the addition of multiplicative white noise, in the sense of Itô, stabilizes the stationary solution $x \equiv 0$, while the Stratonovich interpretation of the noise does not produce any kind of stabilization. In fact, we prove that the model possesses a random attractor and that its Hausdorff dimension is of the same order than the deterministic attractor for the model without noise. The main tool to prove a lower bound on the dimension is the existence of a local unstable invariant manifold.**

I. Introduction

We study a reaction-diffusion equation perturbed by a multiplicative white noise term, first considering this in the Itô sense,

$$dx(t) = \Delta x(t)\, dt + (ax(t) - x(t)^3)\, dt + \sigma x(t)\, dW_t.$$

First, we study the stabilizing effect of the stochastic perturbation on the solution $x = 0$, and show that the interval of stability increases as σ is increased, so that large levels of noise simplify the long-time behaviour. In some sense, we could interpret that the dynamics of the system is substantially reduced to a very basic one driven by the equilibrium of the model.

We can write the Itô equation in the alternative Stratonovich form (Stratonovich [10]),

$$dx(t) = \Delta x(t)\, dt + ((a - \tfrac{1}{2}\sigma^2)x(t) - x(t)^3)\, dt + \sigma x(t) \circ dW_t,$$

which makes the stabilization effect of the noise term explicit, and in the remainder of the work we treat the equation

$$dx(t) = \Delta x(t)\, dt + (\beta x(t) - x(t)^3)\, dt + \sigma x(t) \circ dW_t,$$

investigating its behaviour as β is varied.

To be more precise, let $D \subset R^m$, $m \leq 5$, be an open bounded set with regular boundary, and we consider the following Chafee-Infante reaction-diffusion equation in D

$$du = (\Delta u + \beta u - u^3)dt + \sigma u \circ dW_t, \tag{1}$$

with $u(x,t) = 0$ for $x \in \partial D$, and where $W_{\cdot}(\omega) : \Omega \to C^0(R, R)$ is a one-dimensional Wiener process on the probability space $(\Omega, \mathcal{F}, P)$.

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We rewrite (1) as the following differential equation on $L^2(D)$,

$$du = (-Au + \beta u - u^3)\, dt + \sigma u \circ dW_t \tag{2}$$

where $A = -\Delta$ on D with the appropriate (Dirichlet) boundary conditions. The operator A is positive, linear, self-adjoint and with compact inverse. Thus there exists a sequence $\{\lambda_j\}$ of positive eigenvalues, whose associated eigenfunctions w_j (with $Aw_j = \lambda_j w_j$) form an orthonormal basis for H (e.g. Renardy & Rogers [9]). We order these so that $\lambda_{n+1} \geq \lambda_n$.

A general study of the deterministic reaction-diffusion equation includes an understanding of its global attractor. In this relatively simple case, this attractor consists of the stationary solutions, which are joined by the stable and unstable manifolds associated with them (see Hale [6] or Henry [7] for example). If we wish to continue our investigation of the stochastic version of this PDE, by analogy it seems sensible to investigate whether or not the random dynamical system generated by the model possesses a random attractor. In particular, one can prove that, if $\beta < \lambda_1$, the random attractor is in fact the deterministic point $\{0\}$, whereas if $\beta \geq \lambda_1$ the attractor could be a much more general (random) set.

One of the most interesting properties of certain attractors of deterministic infinite-dimensional dynamical systems is that they are finite-dimensional subsets of the infinite-dimensional phase space (see Temam [13], for example). To try to get a little more information about the possible complexities of our random attractor when $\beta \geq \lambda_1$, we use the method developed by Debussche in [5] to obtain a bound on its Hausdorff dimension. In particular, it holds that if

$$\beta < \frac{1}{d} \sum_{j=1}^{d} \lambda_j,$$

where λ_j are the eigenvalues of the Laplacian arranged in increasing order, then the Hausdorff dimension of the random attractor is bounded by d, P-almost surely. Once again, a little further argument recovers the result that if $\beta < \lambda_1$ then the attractor consists of just one point; but now we can see that increasing β at least allows (within the bound above) for more complexity of the attractor.

However, it is more intriguing to ensure that there exists a lower bound for the dimension since this will ensure that the level of complexity of the random attractor is at least similar to the one in the deterministic case. In this direction, it is proved that in the case $m \leq 5$, there exists a lower bound on the dimension of the random attractor which is of the same order in β as the upper bound, and is the same as that obtained in the deterministic case. Then we show, for $m = 1$, that as β passes through λ_1 from below, the system undergoes a stochastic bifurcation of pitchfork type. Central to this approach is the existence of a random unstable manifold.

II. RANDOM ATTRACTORS

Let us recall the definition of a random dynamical system and a random attractor.

A. Random dynamical systems

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\theta_t : \Omega \to \Omega,\ t \in R\}$ a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t \omega$ is measurable, $\theta_0 = \mathrm{id}$, and $\theta_{t+s} = \theta_t \theta_s$ for all $s, t \in R$. The flow θ_t together with the corresponding probability space, $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in R})$ is called a *(measurable) dynamical system* (see [1], [4], [11]).

A continuous *random dynamical system* (RDS) on a Polish space (X, d) with Borel σ-algebra $\mathcal{B}$ over θ on $(\Omega, \mathcal{F}, P)$ is a measurable map

$$\begin{aligned} \varphi : R^+ \times \Omega \times X &\to X \\ (t, \omega, x) &\mapsto \varphi(t, \omega)x \end{aligned}$$

such that $P - a.s.$

i) $\varphi(0, \omega) = \mathrm{id}$ on X

ii) $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$ for all $t, s \in R^+$ (cocycle property)

iii) $\varphi(t, \omega) : X \to X$ is continuous.

When $\sigma \neq 0$ it is known (Pardoux [8]) that for each $u_0 \in L^2(D)$ and $T > 0$, there exists a unique solution $u(t; x_0)$ of (2), belonging to the spaces $L^2(\Omega \times (0, T); H_0^1(D))$, $L^4(\Omega \times (0, T) \times D)$ and $L^2(\Omega; C(0, T; L^2(D))$. In particular, it follows that the solutions of (2) can be used to generate a random dynamical system if we define $\varphi(t, \omega)u_0 = u(t; \omega, u_0)$, where $u(t; \omega, u_0)$ is the solution of (2) with noise ω and initial condition $u(0) = u_0$.

B. Random attractors

A random set $\mathcal{A}(\omega)$ is said to be a *random attractor* for the RDS φ if

i) $\mathcal{A}(\omega)$ is a random compact set, that is, $P - a.s.$, $\mathcal{A}(\omega)$ is compact and for all $x \in X$, and the map $\omega \mapsto (x, \mathcal{A}(\omega))$ is measurable with respect to $\mathcal{F}$.

ii) $P - a.s.\ \varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega) t \geq 0$ (invariance) and

iii) for all $B \subset X$ bounded (and non-random), $P - a.s.$,

$$\lim_{t \to \infty} (\varphi(t, \theta_{-t}\omega)B, \mathcal{A}(\omega)) = 0,$$

where $(. , .)$ denotes the Hausdorff semidistance in X.

Since $\varphi(t, \theta_{-t}\omega)u_0$ can be interpreted as the position at $t = 0$ of the trajectory which was at u_0 at time $-t$, this *pullback convergence* property is essentially attraction 'from $t = -\infty$'.

It is proved (see Caraballo *et al.* [2]) the existence of a random attractor for our equation. We also showed that if $\beta < \lambda_1$ then $\mathcal{A}(\omega) = \{0\}$, and more generally that if $\beta < \frac{1}{d} \sum_{j=1}^{d} \lambda_j$ then $d_H((\omega)) < d$. Since one can bound $\sum_{j=1}^{d} \lambda_j \leq C d^{(m+2)/m}$, this implies the upper bound

$$d_H(A(\omega)) \leq c_1 \beta^{m/2}, \tag{3}$$

which is an amount of the same order as the deterministic bound, and of course is extremely suggestive of the fact that the attractor becomes more complicated as β increases through λ_1. The confirmation of this point becomes true once we obtain a lower bound for the dimension. This was proved in Caraballo *et al.* [3].

Theorem. *Provided that $m \leq 5$, if $\lambda_n < \beta < \lambda_{n+1}$ then $d_H((\omega)) \geq n$. In particular, the dimension of the attractor satisfies $d_H(\mathcal{A}(\omega)) = O(\beta^{m/2})$.*

Main ideas for the proof. First we truncate the equation, then we prove the existence of an invariant manifold for the truncated equation, and finally, we show that the inertial manifold found in the previous step is in fact part of the unstable set of the origin, within the small neighbourhood on which the truncated and original equations agree.

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